L∞-algebras of Einstein-Cartan-Palatini gravity

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L_∞-Algebras of Einstein–Cartan–Palatini Gravity

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Abstract

We give a detailed account of the cyclic L_∞-algebra formulation of general relativity with cosmological constant in the Einstein–Cartan–Palatini formalism on spacetimes of arbitrary dimension and signature, which encompasses all symmetries, field equations and Noether identities of gravity without matter fields. We present a local formulation as well as a global covariant framework, and an explicit isomorphism between the two L_∞-algebras in the case of parallelizable spacetimes. By duality, we show that our L_∞-algebras describe the complete BV–BRST formulation of Einstein–Cartan–Palatini gravity. We give a general description of how to extend on-shell redundant symmetries in topological gauge theories to off-shell correspondences between symmetries in terms of quasi-isomorphisms of L_∞-algebras. We use this to extend the on-shell equivalence between gravity and Chern–Simons theory in three dimensions to an explicit L_∞-quasi-isomorphism between differential graded Lie algebras which applies off-shell and for degenerate dynamical metrics. In contrast, we show that there is no morphism between the L_∞-algebra underlying gravity and the differential graded Lie algebra governing BF theory in four dimensions.

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1 Introduction

Recent developments in string theory have suggested that the low-energy effective dynamics of closed strings in non-geometric flux compactifications may be governed by noncommutative and even nonassociative deformations of gravity [1–6]. The framework of nonassociative differential geometry was developed in this context in [7–9]. However, the metric aspects of the theory have proved more difficult to develop fully; in particular, a suitable generalization of the Einstein–Hilbert action is not known. On the other hand, noncommutative and nonassociative deformations of gravity are possible to study in the Einstein–Cartan formulation [10, 11], and this is the main motivation behind the present paper: We wish to understand the symmetries of these deformed gravity theories, their field equations and their Lagrangian formulations.

One possible path towards systematically understanding the symmetries and dynamics of noncommutative and nonassociative gravity is through the language of L_∞-algebras. L_∞-algebras are generalizations of differential graded Lie algebras with infinitely-many graded antisymmetric brackets, related to each other by higher homotopy versions of the Jacobi identity. Their first appearence in the physics literature can be traced back to higher spin gauge theories, where closure of the gauge algebra necessitates using field dependent gauge parameters [12]. They first appeared systematically in closed bosonic string field theory, where they govern the "generalized" gauge symmetries and dynamics of the theory [13]: both the gauge transformations and field equations of the theory involve infinitely-many higher brackets of a cyclic L_∞-algebra. They were systematically treated in the mathematics literature [14], where they were shown to be dual to differential graded commutative algebras. In [15] it was suggested that the complete data of classical field theories fit into truncated versions of L_∞-algebras with finitely many non-vanishing brackets, again encoding both gauge transformations and dynamics; the prototypical examples are pure gauge theories such as Yang–Mills theory and Chern–Simons theory. This was shown much earlier by [16] to be a consequence of the duality with the BV–BRST formalism, the details of which were further explained recently by [17]. Indeed, the BV–BRST complex is the familiar physics incarnation of the duality between differential graded commutative algebras and L_∞-algebras: One may directly convert the BV complex of a classical field theory to an L_∞-algebra, and *vice versa*. This explains why the "generalized" gauge symmetries and dynamics of every classical perturbative field theory are organised by an underlying L_∞-algebra structure.[1]

The framework of L_∞-algebras naturally seems to allow the possibility for encoding noncommutativity and nonassociativity, particularly the covariance of curvature fields and closure of the gauge algebras which no longer follow the usual classical rules in general. Indeed, L_∞-algebras are known to play a crucial role in deformation theory, a famous example being Kontsevich's formality theorem in deformation quantization whose proof is based on L_∞-quasi-isomorphisms of differential graded Lie algebras [18]. It was shown in [19, 20] that noncommutative and nonassociative versions of the standard gauge theories fit the same prescription as their classical counterparts, though typically with infinitely-many brackets. This suggests the possibility of encoding nonassociative gravity in the language of L_∞-algebras, a setting where the symmetries and dynamics would become more transparent and perhaps an action functional could even be identified. However, there is even more power in the approach: The L_∞-algebra formulation can also treat non-Lagrangian field theories, which have no action principle.

Classical general relativity on a d-dimensional manifold M with metric g and dynamics governed

[1] This holds only for polynomial field theories whose underlying spaces of fields are vector spaces or affine spaces. Otherwise, one is restricted to the pertubation theory around a classical solution.

by the Einstein–Hilbert action functional

$$S_{\rm EH}(g) = \frac{1}{2\,\kappa^2}\,\int_M (R-2\,\Lambda)\,\sqrt{-g}\,\mathrm{d}^d x \tag{1.1}$$

cannot be directly interpreted as a gauge theory of principal bundle connections. Of course, it does define a 'generalized' gauge theory in a certain extended sense whose symmetries are diffeomorphisms of M, and the corresponding BV formalism can be developed as in e.g. [21]. However, this requires working on the space of non-degenerate metric tensors (as does the very definition (1.1)), which is an open subset of the vector space of all symmetric rank 2 tensors on M, and so does not fit naturally into an L_∞-algebra framework wherein the space of dynamical fields is required to be a vector space. The way around this is to remember that the L_∞-algebra formulation works at the perturbative level and to linearize the theory by expanding the metric g around a chosen background. The space of metric fluctuations h is now a vector space, but the L_∞-algebra formulation will involve infinitely-many brackets of h from the expansion of the Einstein equations coming from (1.1) about the fixed background, as described in [15]; the L_∞-algebra approach to general relativity in this linearized setting is also discussed by [22, 23].

To avoid the introduction of an infinity of brackets, one may instead appeal to the Einstein–Cartan formulation which rewrites general relativity as a gauge theory on a principal bundle over M; the corresponding action is the Palatini action whose definition allows for degenerate configurations, hence having a linear space of fields. This is the theory that we shall work with in this paper; we call it the Einstein–Cartan–Palatini (ECP) formulation of gravity.[2] The purpose of this paper is then twofold. Firstly, we will express ECP gravity in the framework of L_∞-algebras, which should be familiar to experts in the BV formalism, but perhaps not to a broader audience; the BV–BRST formulation in four dimensions is developed in [24] and the L_∞-algebras we present in $d = 4$ may also be obtained by direct dualization, one of which we show explicitly. Working in the framework of L_∞-algebras, including degenerate metrics, allows for inversion up to homotopy of the relevant morphisms, a fact we make use in the main text. The physical (or otherwise) nature of degenerate solutions has been long studied, see for instance [25] [26] [27], but we shall not concern ourselves with this matter in this paper. Secondly, the formalism of this paper is the first step towards setting up a framework for a new approach to noncommutative and nonassociative deformations of gravity, and a larger class of theories satisfying a certain module property. It is in the course of thinking about this latter problem that we realised a complete and explicit account of the first order formalism for general relativity in the framework of L_∞-algebras does not seem to be available in the literature, and in the following we focus on the classical case; its noncommutative and nonassociative deformations will be treated in subsequent papers, see [28] for a glimpse of the advantages of the approach. We translate the ECP formulation of gravity in arbitrary spacetime dimension d into the L_∞-algebra framework, including the dynamics, the local gauge and diffeomorphism symmetries, and the corresponding Noether identities; as we work on a Lie algebraic level, the signature of spacetime has no bearing on our calculations and we shall usually keep it arbitrary. Our formalism sets the stage for many future investigations into the role played by L_∞-algebras in classical general relativity. For instance, deformations of the ECP functional and equivalences to other field theories via L_∞-quasi-isomorphisms could be investigated. Furthermore, one can reinterpret and construct on-shell tree-level graviton scattering amplitudes as brackets for the minimal model corresponding to our ECP L_∞-algebra, along the lines of [23]. Moreover, quasi-isomorphisms have been recently shown to underlie spontaneous symmetry breaking in terms of perturbation theory in gauge theories [29], that is when the underlying L_∞-algebras are interepreted as (derived) tangent complexes around classical solutions. Identifying such equivalences for perturbations of gravity around classical backgrounds, breaking full diffeomorphism symmetry, is certainly of interest.

[2] It is also (perhaps more correctly) known as the Einstein–Cartan–Sciama–Kibble theory.

From the perspective of non-geometric string theory, it is natural to restrict the spacetime manifold M to flat space $\mathbb{R}^d$, hence all bundles considered are trivial. In this paper we will focus mostly on non-covariant (local) gauge transformations, which are applicable only on parallelizable manifolds M. This enables us to make most contact with the existing literature on the ECP approach to general relativity, where global issues are typically ignored. Under some constraints, this is not a big restriction. For example, all orientable three-manifolds are parallelizable. In the initial value problem in general relativity, one usually assumes that the spacetime M is globally hyperbolic. In four dimensions, global hyperbolicity implies $M \simeq \mathbb{R} \times N$, so that if the Cauchy surface N is orientable then the spacetime M is also parallelizable. However, one can also consider more general spacetime manifolds where this approach is insufficient. We shall address this point in the following and demonstrate how to modify the L_∞-algebra formulation to a global covariant structure, which illustrates the power of working in the full L_∞-algebra picture; for example, while the local dynamics of three-dimensional general relativity can be formulated entirely in terms of differential graded Lie algebras, the covariant framework requires extending to the larger category of L_∞-algebras as higher brackets are introduced. The covariant structure we describe is essentially dual to the BV-BRST formalism developed in [24], and in this sense should be viewed as a review, however in the L_∞-algebra picture we clarify the geometric meaning behind the properties of the covariant Lie derivatives and different terms appearing in the BV differential.

Nevertheless, the non-covariant L_∞-algebra structure for parallelizable spacetimes will avoid the global issues involved when twisting the framework to noncommutative principal bundles [28]. The global approach that we present in this paper would be much more involved from the perspective of twist deformation quantization, where one would have to deal with finite noncommutative gauge transformations as putative "transition functions". Furthermore, the noncommutative theory developed in [28] and subsequent papers will be viewed as a low-energy effective theory of gravity which encodes corrections due to noncommutativity. Since this asymptotic expansion of the deformation quantization is formal, and usually only makes sense on affine spaces, global issues are ignored from the outset so that the extraction of explicit first order corrections is immediate.

Summary of results and outline

The main results and outline of the remainder of this paper are as follows. Sections 2 and 3 review the main background material needed in the rest of the paper. In Section 2 we introduce the basic notions surrounding cyclic L_∞-algebras and their morphisms that we will use, and how they completely determine the symmetries and dynamics of classical perturbative field theories with generalized gauge symmetries; we further describe the duality with differential graded commutative algebras which connects the L_∞-algebra formalism with the BV–BRST formalism. In Section 3 we introduce the geometric formulation of Einstein–Cartan–Palatini gravity with cosmological constant Λ in arbitrary spacetime dimension d and signature, including a description of its local gauge and diffeomorphism symmetries, its action functional and field equations, and the corresponding Noether identities.

In Section 4 we consider the L_∞-algebra formulation of a simple class of topological field theories that are related to our gravity theories, and which have gauge and shift symmetries. The L_∞-algebras in these instances are differential graded Lie algebras. By virtue of their topological character, these theories are diffeomorphism-invariant, but diffeomorphisms are redundant symmetries: they are equivalent on-shell to local gauge and shift transformations with field dependent parameters. We demonstrate how to extend this correspondence between symmetries to an *off-shell* equivalence by explicitly constructing a quasi-isomorphism to an extended L_∞-algebra. Working in the larger category of L_∞-algebras is crucial for this equivalence, as the morphism is not a quasi-isomorphism of differential graded Lie algebras. In dimensions $d \geq 4$ these theories also exhibit a simple instance of on-shell higher gauge symmetries. These observations are used in our later attempts to connect ECP theories with topological gauge theories in the L_∞-algebra framework.

In Section 5, we present our main construction of the cyclic L_∞-algebra determining ECP gravity, in any dimension d and spacetime signature. We give an explicit construction of the brackets and the cyclic pairing underlying both local gauge and diffeomorphism symmetries, the field equations and corresponding Noether identities, and the action functional. The structure of the L_∞-algebra is largely dependent on the spacetime dimension d. For instance, only for $d = 3$ is the L_∞-algebra a differential graded Lie algebra. We proceed in Section 6 to review the BV–BRST formalism for ECP gravity in arbitrary dimension and signature, developed recently in [24] for d=4, and show that it is dual to our formulation in terms of L_∞-algebras.

The construction works either locally on M, or globally if the spacetime M is parallelizable. We discuss the incompatibility issues with this construction in the case of non-parallelizable spacetimes where the underlying bundles need not be trivial, and more generally (even when all bundles are trivial) the incompatibility of infinitesimal diffeomorphisms of the physical spacetime with finite gauge transformations. We rectify the problem by defining a 'covariant' version of our L_∞-algebra, dual to the construction of [24], which encompasses the symmetries and dynamics of ECP gravity on general spacetimes with general bundles, and which is compatible with finite gauge transformations. We review the geometric meaning of the "covariant" Lie derivative appearing, which has already been widely used in the first order literature and beyond, and clarify its relation with the new brackets. The covariant framework illustrates the necessity of working in the category of L_∞-algebras: As it always introduces higher brackets, the covariant L_∞-algebra is never a differential graded Lie algebra. It also demonstrates the importance of including Noether identities into the underlying cochain complex in order to capture the covariance of the Euler–Lagrange derivatives of the theory. For parallelizable spacetimes, we present a (strict) isomorphism between the local and covariant L_∞-algebras, dual to the symplectomorphism for four-dimensional gravity in the BV formalism developed in [24].

We conclude by applying our constructions to some lower-dimensional cases of particular interest. In Section 7 we look at three-dimensional gravity, whose underlying ECP L_∞-algebra is a differential graded Lie algebra. In this case, gravity is known to be equivalent to a Chern–Simons gauge theory, where the diffeomorphism symmetry is recovered on-shell by the gauge symmetries of Chern–Simons theory. Using the framework we develop in Section 4, we construct an explicit L_∞-quasi-isomorphism between the differential graded Lie algebras of the two theories, which extends the equivalence both off-shell and to degenerate metrics. This problem is also addressed within the strictly non-degenerate setting using the BV formalism in [30] by providing a (different) symplectomorphism to BF theory. Our result may be viewed as an extension to the degenerate sector. The equivalence is also addressed in [31] using stacks. Finally, in Section 8 we consider four-dimensional gravity, whose underlying ECP L_∞-algebra is no longer a differential graded Lie algebra. Abstract strictification theorems [32] imply that any L_∞-algebra is quasi-isomorphic to a differential graded Lie algebra, though in practise the construction of the quasi-isomorphism is very difficult and not very convenient to make explicit. Applying this to four-dimensional gravity, the strictification of its ECP L_∞-algebra could possibly correspond to some deformation of BF theory in four dimensions, as reviewed in [33]. Indeed, we show that there cannot exist any L_∞-morphism between the L_∞-algebra underlying four-dimensional gravity and the differential graded Lie algebra of BF theory.

Two appendices at the end of the paper contain illustrative examples of the details involved in the long cumbersome calculations required to check the various homotopy relations for the ECP L_∞-algebras and the L_∞-morphisms that we present in the main text, and to establish the duality with the BV–BRST complex of ECP gravity. In view of the duality, one may start from the BV-BRST framework and derive the algebras we present. However, the L_∞-algebras presented in the text have been constructed using the bootstrap approach, unless otherwise explicitly noted. The duality then served as an additional consistency check with known results. Indeed, this work serves as a starting point in twisting the corresponding L_∞-algebras and one needs not be fluent in BV–BRST to follow the procedure [28], which applies to a general class of theories whose L_∞-algebras are modules of a certain Hopf algebra. For this reason, we present the material starting directly from the L_∞-algebra picture while making contact with BV–BRST later on. We hope these detailed calculations are instructive and provide a useful reference for further investigations into the L_∞-algebra picture of gravity. They are utilized extensively in [28] and subsequent papers.

2 L_∞-algebras, differential graded algebras and field theory

In this section we review the required algebraic constructions, and their applications to classical field theories, that are needed in this paper.

2.1 L_∞-algebras

We start by introducing notions of L_∞-algebras, which form the central concept in this paper.

Brackets and homotopy relations

An L_∞-algebra is a $\mathbb{Z}$-graded vector space $V = \bigoplus_{k\in\mathbb{Z}} V_k$ equipped with graded antisymmetric multilinear maps

$$\ell_n : \textstyle\bigwedge^n V \longrightarrow V \ , \quad v_1\wedge\cdots\wedge v_n \longmapsto \ell_n(v_1,\dots,v_n)$$

for each $n\geq 1$, which we call n-brackets. The graded antisymmetry translates to

$$\ell_n(\dots,v,v',\dots) = -(-1)^{|v|\,|v'|}\,\ell_n(\dots,v',v,\dots) \ , \tag{2.1}$$

where we denote the degree of a homogeneous element $v\in V$ by $|v|$. The n-bracket is a map of degree $|\ell_n| = 2-n$, that is

$$\big|\ell_n(v_1,\dots,v_n)\big| = 2-n+\sum_{j=1}^n |v_j| \ .$$

The n-brackets ℓ_n are required to fulfill infinitely many identities $\mathcal{J}_n(v_1,\dots,v_n)=0$ for each $n\geq 1$, called homotopy relations, with

$$\begin{aligned}\mathcal{J}_n(v_1,\dots,v_n) := \sum_{i=1}^n(-1)^{i\,(n-i)}\sum_{\sigma\in\mathrm{Sh}_{i,n-i}}&\chi(\sigma;v_1,\dots,v_n)\\ &\times\ell_{n+1-i}\big(\ell_i(v_{\sigma(1)},\dots,v_{\sigma(i)}),v_{\sigma(i+1)},\dots,v_{\sigma(n)}\big) \ ,\end{aligned} \tag{2.2}$$

where, for each $i=1,\dots,n$, the second sum runs over $(i,n-i)$-shuffled permutations $\sigma\in S_n$ of degree n which are restricted as

$$\sigma(1)<\cdots<\sigma(i) \qquad \text{and} \qquad \sigma(i+1)<\cdots<\sigma(n) \ .$$

The Koszul sign $\chi(\sigma;v_1,\dots,v_n)=\pm 1$ is determined from the grading by

$$v_{\sigma(1)}\wedge\cdots\wedge v_{\sigma(n)} = \chi(\sigma;v_1,\dots,v_n)\ v_1\wedge\cdots\wedge v_n \ .$$

For example, the first three identities are given by

$$
\begin{aligned}
0 &= \mathcal{J}_1(v) = \ell_1\big(\ell_1(v)\big) \ , \\
0 &= \mathcal{J}_2(v_1,v_2) = \ell_1\big(\ell_2(v_1,v_2)\big) - \ell_2\big(\ell_1(v_1),v_2\big) - (-1)^{|v_1|}\,\ell_2\big(v_1,\ell_1(v_2)\big) \ , \\
0 &= \mathcal{J}_3(v_1,v_2,v_3) \\
&= \ell_1\big(\ell_3(v_1,v_2,v_3)\big) \\
&\quad + \ell_3\big(\ell_1(v_1),v_2,v_3\big) + (-1)^{|v_1|}\,\ell_3\big(v_1,\ell_1(v_2),v_3\big) + (-1)^{|v_1|+|v_2|}\,\ell_3\big(v_1,v_2,\ell_1(v_3)\big) \\
&\quad + \ell_2\big(\ell_2(v_1,v_2),v_3\big) + (-1)^{(|v_1|+|v_2|)\,|v_3|}\,\ell_2\big(\ell_2(v_3,v_1),v_2\big) + (-1)^{(|v_2|+|v_3|)\,|v_1|}\,\ell_2\big(\ell_2(v_2,v_3),v_1\big) \ .
\end{aligned} \tag{2.3}
$$

The first identity states that the map $\ell_1 : V \to V$ is a differential making V into a cochain complex

$$
\cdots \xrightarrow{\ \ell_1\ } V_k \xrightarrow{\ \ell_1\ } V_{k+1} \xrightarrow{\ \ell_1\ } \cdots \ .
$$

The second identity states that ℓ_1 is a graded derivation with respect to the 2-bracket ℓ_2, that is, $\ell_2 : V_k \wedge V_l \to V_{k+l}$ is a cochain map. For $\ell_3 = 0$ the third identity is just the graded Jacobi identity for the 2-bracket ℓ_2, while for $\ell_2 = 0$ it gives the graded Leibniz rule for the differential ℓ_1 with respect to the 3-bracket ℓ_3; in general it expresses the coherence condition that makes the Jacobiator for ℓ_2 on $V_k \wedge V_l \wedge V_m \to V_{k+l+m-1}$ a cochain homotopy, that is, the Jacobi identity is violated by a homotopy. In this sense L_∞-algebras are (strong) homotopy deformations of differential graded Lie algebras which are the special cases where the ternary and all higher brackets vanish: $\ell_n = 0$ for all $n \geq 3$. In general, the homotopy relations for $n \geq 3$ are generalized Jacobi identities; for later use, we note that the identity $\mathcal{J}_4 = 0$ is given by

$$
\begin{aligned}
\mathcal{J}_4(v_1,v_2,v_3,v_4) = {}& \ell_1\big(\ell_4(v_1,v_2,v_3,v_4)\big) \\
& - \ell_4\big(\ell_1(v_1),v_2,v_3,v_4\big) - (-1)^{|v_1|}\,\ell_4\big(v_1,\ell_1(v_2),v_3,v_4\big) \\
& - (-1)^{|v_1|+|v_2|}\,\ell_4\big(v_1,v_2,\ell_1(v_3),v_4\big) - (-1)^{|v_1|+|v_2|+|v_3|}\,\ell_4\big(v_1,v_2,v_3,\ell_1(v_4)\big) \\
& - \ell_2\big(\ell_3(v_1,v_2,v_3),v_4\big) + (-1)^{|v_3|\,|v_4|}\,\ell_2\big(\ell_3(v_1,v_2,v_4),v_3\big) \\
& + (-1)^{(1+|v_1|)\,|v_2|}\,\ell_2\big(v_2,\ell_3(v_1,v_3,v_4)\big) - (-1)^{|v_1|}\,\ell_2\big(v_1,\ell_3(v_2,v_3,v_4)\big) \\
& + \ell_3\big(\ell_2(v_1,v_2),v_3,v_4\big) - (-1)^{|v_2|\,|v_3|}\,\ell_3\big(\ell_2(v_1,v_3),v_2,v_4\big) \\
& + (-1)^{(|v_2|+|v_3|)\,|v_4|}\,\ell_3\big(\ell_2(v_1,v_4),v_2,v_3\big) + \ell_3\big(v_1,\ell_2(v_2,v_3),v_4\big) \\
& + (-1)^{|v_3|\,|v_4|}\,\ell_3\big(v_1,\ell_2(v_2,v_4),v_3\big) + \ell_3\big(v_1,v_2,\ell_2(v_3,v_4)\big) \ .
\end{aligned} \tag{2.4}
$$

L_∞-morphisms

The natural notion of a homomorphism from an L_∞-algebra $(V,\{\ell_n\})$ to another L_∞-algebra $(V',\{\ell'_n\})$ consists of a collection of multilinear graded antisymmetric maps

$$
\psi_n : \textstyle\bigwedge^n V \longrightarrow V' \ , \quad v_1 \wedge \cdots \wedge v_n \longmapsto \psi_n(v_1,\dots,v_n)
$$

of degree $|\psi_n| = 1-n$ for each $n \geq 1$, which satisfies an appropriate identity intertwining the two sets of brackets. The required identity is specified through the somewhat cumbersome relations

$$
\begin{aligned}
&\sum_{i=1}^n (-1)^{i\,(n-i)} \sum_{\sigma\in \mathrm{Sh}_{i,n-i}} \chi(\sigma;v_1,\dots,v_n)\ \psi_{n+1-i}\big(\ell_i(v_{\sigma(1)},\dots,v_{\sigma(i)}),v_{\sigma(i+1)},\dots,v_{\sigma(n)}\big) \\
&\qquad = \sum_{k=1}^n \frac{1}{k!}\,(-1)^{\frac12\,k\,(k-1)} \sum_{i_1+\cdots+i_k=n} \ \sum_{\sigma\in\mathrm{Sh}_{i_1,\dots,i_k}} (-1)^{\mathcal{Z}(\sigma;v_1,\dots,v_n)}\,\chi(\sigma;v_1,\dots,v_n) \\
&\qquad\qquad \times \ell'_k\big(\psi_{i_1}(v_{\sigma(1)},\dots,v_{\sigma(i_1)}),\dots,\psi_{i_k}(v_{\sigma(n-i_k+1)},\dots,v_{\sigma(n)})\big) \ ,
\end{aligned} \tag{2.5}
$$

where, for each $k = 1, \dots, n$, the fifth sum runs over $(i_1, \dots, i_k)$-shuffled permutations $\sigma \in S_n$ which preserve the ordering within each block of length $i_1, \dots, i_k$ of the partition of $n = i_1 + \cdots + i_k$. The additional sign factor is given by

$$\mathcal{Z}(\sigma; v_1, \dots, v_n) = \sum_{j=1}^{k-1} (k-j)\, i_j + \sum_{j=2}^{k} (1 - i_j) \sum_{l=1}^{i_1+\cdots+i_{j-1}} \big|v_{\sigma(l)}\big|$$

for $\sigma \in \mathrm{Sh}_{i_1,\dots,i_k}$. Such a homomorphism is called an L_∞-morphism. Notice that the left-hand side of (2.5) is formally identical to the homotopy relations (2.2) with ℓ_{n+1-i} replaced by ψ_{n+1-i}.

The first condition for $n = 1$ (internal degree 1) is given by

$$\psi_1\big(\ell_1(v)\big) = \ell_1'\big(\psi_1(v)\big) \ ,$$

which states that the map $\psi_1 : V \to V'$ is a cochain map with respect to the $n = 1$ differentials, that is, it defines a map of the cochain complexes underlying the L_∞-algebras that acts degreewise as

$$\begin{tikzcd} \cdots \arrow[r, "\ell_1"] & V_k \arrow[r, "\ell_1"] \arrow[d, "\psi_1"'] & V_{k+1} \arrow[r, "\ell_1"] \arrow[d, "\psi_1"'] & \cdots \\ \cdots \arrow[r, "\ell_1'"] & V_k' \arrow[r, "\ell_1'"] & V_{k+1}' \arrow[r, "\ell_1'"] & \cdots \end{tikzcd}$$

and so descends to a homomorphism of the corresponding cohomology groups

$$\psi_{1*} : H^\bullet(V, \ell_1) \longrightarrow H^\bullet(V', \ell_1') \ . \tag{2.6}$$

The second condition for $n = 2$ (internal degree 0) reads

$$\psi_1\big(\ell_2(v_1, v_2)\big) - \ell_2'\big(\psi_1(v_1), \psi_1(v_2)\big) = \ell_1'\big(\psi_2(v_1, v_2)\big) + \psi_2\big(\ell_1(v_1), v_2\big) + (-1)^{|v_1|}\, \psi_2\big(v_1, \ell_1(v_2)\big) \ ,$$

which means that ψ_1 preserves the 2-brackets up to a homotopy given by ψ_2. In particular, if $\psi_n = 0$ for all $n \geq 2$, then ψ_1 generalizes a homomorphism of differential graded Lie algebras. On the other hand, even if the underlying L_∞-algebras are differential graded Lie algebras, an L_∞-morphism is not generally a morphism of differential graded Lie algebras. The third condition for $n = 3$ (internal degree -1) is

$$\begin{aligned} &\psi_3\big(\ell_1(v_1), v_2, v_3\big) + (-1)^{|v_1|}\, \psi_3\big(v_1, \ell_1(v_2), v_3\big) + (-1)^{|v_1|+|v_2|}\, \psi_3\big(v_1, v_2, \ell_1(v_3)\big) - \ell_1'\big(\psi_3(v_1, v_2, v_3)\big) \\ &\qquad + \psi_1\big(\ell_3(v_1, v_2, v_3)\big) - \ell_3'\big(\psi_1(v_1), \psi_1(v_2), \psi_1(v_3)\big) \\ &\quad = (-1)^{|v_1|}\, \ell_2'\big(\psi_1(v_1), \psi_2(v_2, v_3)\big) - (-1)^{|v_2|\,(1+|v_1|)}\, \ell_2'\big(\psi_1(v_2), \psi_2(v_1, v_3)\big) \\ &\qquad\qquad + (-1)^{|v_3|\,(1+|v_1|+|v_2|)}\, \ell_2'\big(\psi_1(v_3), \psi_2(v_1, v_2)\big) \\ &\qquad - \psi_2\big(\ell_2(v_1, v_2), v_3\big) - (-1)^{(|v_1|+|v_2|)\,|v_3|}\, \psi_2\big(\ell_2(v_3, v_1), v_2\big) - (-1)^{(|v_2|+|v_3|)\,|v_1|}\, \psi_2\big(\ell_2(v_2, v_3), v_1\big) \ . \end{aligned}$$

In general, an L_∞-morphism preserves the n-brackets up to homotopy.

From the perspective of the underlying cochain complexes, the natural notion of isomorphism between L_∞-algebras would be an L_∞-morphism whose induced map (2.6) is an isomorphism on the cohomology of the complexes; in this case we call the collection $\{\psi_n\}$ an L_∞-quasi-isomorphism and say that the L_∞-algebras are quasi-isomorphic. Quasi-isomorphism defines an equivalence relation on the broader set of all L_∞-algebras [18], in contrast to the category of differential graded Lie algebras where not every quasi-isomorphism has a homotopy inverse. A stronger notion demands that the degree 0 cochain map $\psi_1 : V \to V'$ itself is an isomorphism of the underlying vector spaces; in this case the collection $\{\psi_n\}$ is called an L_∞-isomorphism and the L_∞-algebras are said to be isomorphic. From an L_∞-isomorphism one can reconstruct all brackets of one L_∞-algebra from the brackets of the other by using the relations (2.5) if the inverse ψ_1^{-1} is known explicitly. Both notions of isomorphism between L_∞-algebras will play a role in this paper.

Cyclic pairings

We will be particularly interested in the case where an L_∞-algebra $(V,\{\ell_n\})$ is further endowed with a graded symmetric non-degenerate bilinear pairing $\langle -,-\rangle : V\otimes V\to\mathbb{R}$ which is cyclic in the sense that

$$\langle v_0,\ell_n(v_1,v_2,\dots,v_n)\rangle = (-1)^{n+(|v_0|+|v_n|)\,n+|v_n|\,\sum_{i=0}^{n-1}|v_i|}\,\langle v_n,\ell_n(v_0,v_1,\dots,v_{n-1})\rangle$$

for all $n\geq 1$. This is the natural notion of an inner product on an L_∞-algebra, and if such a pairing exists the resulting algebraic structure is called a cyclic L_∞-algebra. If in addition the pairing is odd, say of degree $-p$, then the only non-vanishing pairings are $\langle -,-\rangle : V_k\otimes V_{p-k}\to\mathbb{R}$ for $2k<p$ and the cyclicity condition simplifies to

$$\langle \ell_n(v_0,v_1,\dots,v_{n-1}),v_n\rangle = (-1)^{(|v_0|+1)\,n}\,\langle v_0,\ell_n(v_1,\dots,v_n)\rangle\ .$$

Given L_∞-algebras $(V,\{\ell_n\})$ and $(V',\{\ell'_n\})$ which are endowed with cyclic pairings $\langle -,-\rangle$ and $\langle -,-\rangle'$, then an L_∞-morphism $\{\psi_n\}$ between them is cyclic if it additionally preserves the pairing in the sense that

$$\langle \psi_1(v_1),\psi_1(v_2)\rangle' = \langle v_1,v_2\rangle\ ,$$

and

$$\sum_{i=1}^{n-1}(-1)^{i-1+(n-i-1)\,\sum_{j=1}^{i}|v_j|}\,\langle \psi_i(v_1,\dots,v_i),\psi_{n-i}(v_{i+1},\dots,v_n)\rangle' = 0\ , \tag{2.7}$$

for all $n\geq 3$ and $v_1,\dots,v_n\in V$.

2.2 Differential graded commutative algebras

There is a duality between semifree differential graded commutative algebras and L_∞-algebras of finite type, which we describe following the well-known mathematical treatment, see e.g. [14, 34]. A differential graded commutative algebra is a graded commutative algebra $A=\bigoplus_{k\in\mathbb{Z}}A_k$, whose multiplication we denote by $\cdot$, which is endowed with a differential of degree 1 which is a graded derivation, that is, a linear map $\mathrm{d}:A_k\to A_{k+1}$ such that $\mathrm{d}^2=0$ and

$$\mathrm{d}(a\cdot b) = \mathrm{d}(a)\cdot b + (-1)^{|a|}\,a\cdot\mathrm{d}(b)$$

for all $a,b\in A$ with a homogeneous. A graded commutative algebra is said to be of finite type if it is degreewise finite-dimensional; it is called semifree if it is isomorphic to the symmetric tensor algebra of a graded vector space. Semifree differential graded commutative algebras of finite type are in one-to-one correspondence with L_∞-algebras of finite type [14, 34]. We briefly spell out one direction of this correspondence which will be of use later on.

Let $V=\bigoplus_{k\in\mathbb{Z}}V_k$ be a graded vector space. Consider its suspension which is the degree shifted graded space $\mathscr{F}:=V[1]$, that is, $\mathscr{F}_k=V_{k+1}$, and note that the two are related by the trivial suspension isomorphism which decreases the grading by 1:

$$s:V\longrightarrow\mathscr{F}\ ,\quad v\longmapsto {}^s v$$

with $|{}^s v|=|v|-1$.[3] This induces an isomorphism of antisymmetric and symmetric tensor algebras respectively, which is given by

$$s^{\otimes n}:{\textstyle\bigwedge}^n V\longrightarrow{\textstyle\bigodot}^n\mathscr{F}\ ,\quad v_1\wedge\cdots\wedge v_n\longmapsto(-1)^{\sum_{j=1}^{n-1}(n-j)\,|v_j|}\ {}^s v_1\odot\cdots\odot{}^s v_n$$

[3] One can think of the suspension $\mathscr{F}=V[1]$ as the tensor product $\mathbb{R}s\otimes V$ with elements ${}^s v=s\otimes v$, where s is a fixed degree 1 element which has a dual s^{-1} with $s^{-1}\,s=1=-s\,s^{-1}$.

on each tensor power for $n \geq 1$ and on homogeneous elements, and extended linearly.

Using these trivial isomorphisms, one may equivalently identify an L_∞-algebra structure on V as a coderivation D of degree 1 on $\bigodot^\bullet \mathscr{F}$ viewed as a free cocommutative coalgebra,[4] such that $\mathrm{D}^2 = 0$. Explicitly, if $\Delta_{\mathscr{F}} : \bigodot^\bullet \mathscr{F} \to \bigodot^\bullet \mathscr{F} \otimes \bigodot^\bullet \mathscr{F}$ is the free coproduct then

$$\Delta_{\mathscr{F}} \circ \mathrm{D} = (\mathrm{D} \otimes 1 + 1 \otimes \mathrm{D}) \circ \Delta_{\mathscr{F}} \ ,$$

which implies that the coderivation $\mathrm{D} : \bigodot^\bullet \mathscr{F} \to \bigodot^\bullet \mathscr{F}$ is completely determined by its image in $\mathscr{F}$ according to the decomposition

$$\mathrm{pr}_{\mathscr{F}} \circ \mathrm{D} = \sum_{n=1}^{\infty} \mathrm{D}_n$$

with degree 1 component maps $\mathrm{D}_n : \bigodot^n \mathscr{F} \to \mathscr{F}$, where $\mathrm{pr}_{\mathscr{F}} : \bigodot^\bullet \mathscr{F} \to \mathscr{F}$ is the projection to $\mathscr{F}$. Then the relation to the graded antisymmetric n-brackets defined on V, $\ell_n : \bigwedge^n V \to V$, is given by

$$\ell_n := s^{-1} \circ \mathrm{D}_n \circ s^{\otimes n} : {\textstyle\bigwedge^n} V \xrightarrow{s^{\otimes n}} {\textstyle\bigodot^n} \mathscr{F} \xrightarrow{\mathrm{D}_n} \mathscr{F} \xrightarrow{s^{-1}} V \ . \tag{2.8}$$

The homotopy relations are then equivalent to $\mathrm{D}^2 = 0$.

This coalgebra picture is arguably harder to work with: its main advantage is that the homotopy relations become simple and natural, and in principle easier to check. Moreover, when V is of finite type one may unambiguously pass to the dual algebra $\bigodot^\bullet \mathscr{F}^*$ which is a graded commutative algebra under the usual symmetric tensor product. The coderivation then dualizes to a graded derivation

$$Q := \mathrm{D}^* : {\textstyle\bigodot^\bullet} \mathscr{F}^* \longrightarrow {\textstyle\bigodot^\bullet} \mathscr{F}^*$$

of degree 1, such that $Q^2 = 0$ if and only if $\mathrm{D}^2 = 0$. It follows that an L_∞-algebra structure on a graded vector space of finite type is equivalent to a differential derivation of the symmetric algebra of its suspended dual vector space.

Another advantage of the coalgebra formulation is that the notion of an L_∞-morphism becomes more natural and transparent. A morphism between two L_∞-algebras determined by codifferential coalgebras $(\bigodot^\bullet \mathscr{F}, \mathrm{D})$ and $(\bigodot^\bullet \mathscr{F}', \mathrm{D}')$ is then given by a cohomomorphism of codifferential coalgebras:

$$\Psi : \big({\textstyle\bigodot^\bullet} \mathscr{F}, \mathrm{D}\big) \longrightarrow \big({\textstyle\bigodot^\bullet} \mathscr{F}', \mathrm{D}'\big) \ ,$$

that is, a degree 0 cohomomorphism of the underlying free cocommutative coalgebras which intertwines the codifferentials: $\Psi \circ \mathrm{D} = \mathrm{D}' \circ \Psi$; with the same sign conventions for the components ψ_n of Ψ as used in (2.8), this single cohomomorphism is equivalent to the collection of maps $\{\psi_n\}$ defining the L_∞-morphism. The dual map $\Psi^* : \bigodot^\bullet \mathscr{F}^* \to \bigodot^\bullet \mathscr{F}'^*$ is a degree-preserving algebra homomorphism which intertwines the corresponding derivations:

$$Q \circ \Psi^* = \Psi^* \circ Q' \ .$$

An L_∞-quasi-isomorphism in this picture is then naturally an algebra homomorphism which is an isomorphism between the degree 0 cohomology groups of the differentials Q and Q', whereas an L_∞-isomorphism is equivalently a coalgebra isomorphism of the corresponding coalgebras.

Finally, a graded symmetric non-degenerate pairing $\langle -,- \rangle$ on V translates into a graded antisymmetric pairing on the suspension $\mathscr{F} := V[1]$ defined by $\langle -,- \rangle \circ (s^{-1} \otimes s^{-1})$. Since it is non-degenerate it qualifies as a graded symplectic pairing on $\mathscr{F}$. This then canonically induces a

[4] One should actually consider the reduced tensor coalgebra, that is, excluding the zeroth tensor power which is a copy of $\mathbb{R}$.

"constant" graded symplectic two-form $\omega \in \Omega^2(\mathscr{F})$, which enables one to thus view $\mathscr{F}$ not only as a graded symplectic vector space but also as a graded symplectic manifold. Cyclicity is then equivalent to Q-invariance of ω [17]. To get an intuition for the condition (2.7), which is a shifted version of the condition from [35], consider the case where $\mathscr{F}$ is concentrated in degree 0. Then by restricting a coalgebra morphism $\Psi : \bigodot^\bullet \mathscr{F} \to \bigodot^\bullet \mathscr{F}'$ to the diagonal of $\bigodot^\bullet \mathscr{F}$, the condition that the non-linear smooth map $\mathrm{pr}_{\mathscr{F}'} \circ \Psi_{|_{\mathscr{F}}} : \mathscr{F} \to \mathscr{F}'$ is a symplectomorphism (of manifolds) is equivalent to the cyclicity conditions of Section 2.1.

2.3 Generalized gauge field theories

We will now sketch how the algebraic constructions of this section find natural applications to the treatment of classical field theories. The description we give is meant to provide a small bridge between the physics community interested in bootstrapping L_∞-algebras, the dual BV–BRST construction and the properly rigorous treatment in terms of derived geometry, where the end product is identified as the tangent complex to a derived stack (at the trivial solution) [36]. The procedure outlined should be rather viewed as an algorithm in identifying the spaces and brackets of the algebras in question.

Geometric formulation

In the physical applications of relevance to this paper, the kinematical data of a (classical) field theory with generalized gauge symmetries on an oriented d-dimensional manifold M[5] are encoded in two vector spaces V_0 and V_1, which are respectively the vector spaces of (infinitesimal) gauge parameters and dynamical fields, being typically sections of vector bundles over M. That is, we shall assume $V_0 := \Omega^l(M, \mathcal{C})$ and $V_1 := \Omega^k(M, \mathcal{V})$ where $\mathcal{V}, \mathcal{C}$ are vector bundles over M and $0 \leq l, k \leq d$.[6] The gauge transformations acting on the dynamical fields generate a distribution $\mathscr{D}$ on V_1, which may not be necessarily involutive on the whole of V_1. By 'generalized' gauge transformations we mean that we include symmetries which are not restricted to vertical automorphisms of principal bundles, and so go beyond the usual realm of standard gauge theory. We shall also supplement the picture with the vector spaces $V_2 := \Omega^{d-k}(M, \check{\mathcal{V}})$ and $V_3 := \Omega^{d-l}(M, \check{\mathcal{C}})$, where $\check{\mathcal{V}}, \check{\mathcal{C}}$ denote the dual bundles. The physical content of these spaces will be explained below. In practice it is often convenient to identify the internal dual bundles $\check{\mathcal{V}}, \check{\mathcal{C}}$ with certain isomorphic bundles through internal non-degenerate pairings entering the definition of field theories; we shall see this explicitly in the theories considered in later sections.

The dynamics of the field theory is specified by an action functional $S : V_1 \to \mathbb{R}$, which is a local function of the fields and their jets that is gauge-invariant: $\delta_\lambda S = 0$ for all $\lambda \in V_0$, where $\delta_\lambda = \delta \circ \iota_\lambda + \iota_\lambda \circ \delta$ with δ the exterior derivative on V_1 and ι_λ the contraction with $\lambda \in \Gamma(\mathscr{D})$[7]. Its variation δS is a section of the cotangent bundle of V_1 whose fibers we shall consider of degree 2, that is $\delta S : V_1 \to T^*[-3]V_1$, where $[k]$ denotes a degree shift by $k \in \mathbb{Z}$. Since V_1 is a vector space, we may define its cotangent bundle as the product of V_1 with its dual, for which the correct model in this infinite-dimensional setting is precisely V_2 defined above. That is

$$T^*[-3]V_1 := V_1 \times V_2 \ . \tag{2.9}$$

[5] The discussion applies for non-orientable manifolds, but the pairings appearing in the following should instead map into the density bundle over M.

[6] In the case where V_1 is an affine space, such as a space of connections, the following discussion is trivially modified by fixing a reference element.

[7] We abuse notation slightly and identify the gauge parameter with the vector field it generates on V_1.

In more detail, the variation of the action functional $\delta S : V_1 \to T^*[-3]V_1$, restricted to sections of compact support, is a map

$$\delta S : V_1 \longrightarrow V_1 \times V_2$$

acting by $\delta S_{|A}(\delta A) = \langle \delta A, \mathcal{F}(A)\rangle$ on tangent vectors $\delta A \in T_A V_1 \simeq V_1$ at a point $A \in V_1$ in the space of fields. The non-degenerate pairing $\langle -,-\rangle : V_1 \times V_2 \to \mathbb{R}$ appearing is the natural pairing, that is, using the duality of the internal bundles and wedging the spacetime form parts to get a top form, following with an integration over M, while $\mathcal{F}(A) \in V_2$ denotes the Euler–Lagrange derivatives of the functional S. Using the above interpretation, we say V_2 is the "space of field equations". The classical solution space, i.e. "on-shell" configurations, is defined as those $A \in V_1$ such that $\mathcal{F}(A) = 0$. Thus the field equations are enforced by intersecting the image of δS with the image of the zero section of T^*V_1, which defines the critical Euler–Lagrange locus of the action functional S; the distribution $\mathscr{D}$ is involutive on this locus, that is, the gauge transformations close on-shell.

The natural pairing extends similarly to $T^*[-3]V_0 := V_0 \times V_3$, where now one may use it to prove Noether's second theorem: For all $\lambda \in V_0$, gauge-invariance of the action functional implies

$$\delta_\lambda S = \langle \delta_\lambda A, \mathcal{F}(A)\rangle = 0 \ ,$$

so that taking the Sturm–Liouville adjoint d_A, with respect to the pairing, of δ_λ viewed as a differential operator acting on a gauge parameter λ of compact support amounts to $\langle \lambda, \mathsf{d}_A \mathcal{F}(A)\rangle = 0$. Here d_A acts as a local differential operator on V_2, which may depend on the fields, and its image is valued in V_3. By non-degeneracy of the extended pairing, the 'Bianchi identities' $\mathsf{d}_A\mathcal{F}(A) = 0$, expressing local differential relations among the Euler–Lagrange derivatives for any infinitesimal local symmetry $\delta_\lambda A$, hold *off-shell*, which is simply a reformulation of Noether's second theorem. Given the above interpretation, we say V_3 is the "space of Noether identities". The converse of Noether's second theorem is a means of recovering gauge symmetries of an action functional S which may be unknown *a priori*.[8]

From this geometric perspective, a classical generalized gauge theory is completely determined by the moduli space $\mathscr{M}$ of its Euler–Lagrange locus $\mathcal{F}(A) = 0$ modulo gauge transformations. Two gauge theories are then physically equivalent, in the sense that there is a bijection between their physical states, if the corresponding moduli spaces of solutions to the field equations are isomorphic: $\mathscr{M} \simeq \mathscr{M}'$. In the case of reducible symmetries[9], the graded vector space may be extended by adjoining V_{-1} containing the "higher" gauge parameters and V_4 its dual containing the "higher" Noether identities in a similar vein as above. Higher level reducibility parameters and their duals may also be added, if they occur. Classically this augmentation does not offer much new information, however it is essential in the dual (quantum) BV-BRST formulation where one is interested in resolving degeneracies for the purpose of path integral methods.

L_∞-algebra formulation

We can translate this geometric picture into the structure of a four-term L_∞-algebra by linearizing the gauge transformations, field equations and Noether identities to obtain the cochain complex

$$V_0 \xrightarrow{\ \ell_1\ } V_1 \xrightarrow{\ \ell_1\ } V_2 \xrightarrow{\ \ell_1\ } V_3 \tag{2.10}$$

corresponding to the underlying graded vector space $V := T^\star[-3]\big(V_0 \oplus V_1\big)$, that is

$$V = V_0 \oplus V_1 \oplus V_2 \oplus V_3 \ .$$

[8]See e.g. [37] for the proofs and comparisons of Noether's first and second theorems, and [38] for their relation to gauge symmetries in physics.

[9]For example, in the case where symmetries arise from a group action which is not free, i.e. has non-trivial stabilisers.

We then equip this complex with suitable higher brackets corresponding to the non-linear parts of the theory subject to the homotopy relations in order the recover the full symmetries and dynamics of the generalized gauge theory.

Given $\lambda \in V_0$ and $A \in V_1$, the gauge variations are encoded as the maps $A \mapsto A + \delta_\lambda A$ where

$$\delta_\lambda A = \sum_{n=0}^{\infty} \frac{1}{n!}\, (-1)^{\frac{1}{2}\, n\, (n-1)}\, \ell_{n+1}(\lambda, A, \dots, A) \ \in \ V_1 \ , \tag{2.11}$$

where the brackets involve n insertions of the field A. The Euler–Lagrange derivatives are encoded as

$$\mathcal{F}(A) = \sum_{n=1}^{\infty} \frac{1}{n!}\, (-1)^{\frac{1}{2}\, n\, (n-1)}\, \ell_{n}(A, \dots, A) \ \in \ V_2 \ , \tag{2.12}$$

with the covariant gauge variations

$$\delta_\lambda \mathcal{F} = \sum_{n=0}^{\infty} \frac{1}{n!}\, (-1)^{\frac{1}{2}\, n\, (n-1)}\, \ell_{n+2}(\lambda, \mathcal{F}, A, \dots, A) \ \in \ V_2 \ . \tag{2.13}$$

We define successive applications of gauge variations by

$$(\delta_{\lambda_1}\, \delta_{\lambda_2})A := \sum_{n=0}^{\infty} \frac{1}{n!}\, (-1)^{\frac{1}{2}\, n\, (n+1)}\, \ell_{n+2}(\lambda_2, \delta_{\lambda_1} A, A, \dots, A) \ .$$

The closure relation for the gauge algebra then has the form

$$[\delta_{\lambda_1}, \delta_{\lambda_2}]A = \delta_{[\![\lambda_1, \lambda_2]\!]_A} A + \Delta_{\lambda_1, \lambda_2} A \ , \tag{2.14}$$

where

$$[\![\lambda_1, \lambda_2]\!]_A = -\sum_{n=0}^{\infty} \frac{1}{n!}\, (-1)^{\frac{1}{2}\, n\, (n-1)}\, \ell_{n+2}(\lambda_1, \lambda_2, A, \dots, A) \ \in \ V_0 \ ,$$

and

$$\Delta_{\lambda_1, \lambda_2} A = \sum_{n=0}^{\infty} \frac{1}{n!}\, (-1)^{\frac{1}{2}\, (n-2)\, (n-3)}\, \ell_{n+3}(\lambda_1, \lambda_2, \mathcal{F}, A, \dots, A) \ \in \ V_1 \ .$$

The distribution $\mathscr{D} \subset TV_1$ spanned by the gauge parameters is involutive on-shell, that is, when $\mathcal{F}(A) = 0$, and on this locus the gauge algebra generally depends on the fields A. The homotopy relations guarantee that the Jacobi identity is generally satisfied for any triple of maps δ_{λ_1}, δ_{λ_2} and δ_{λ_3}. The Noether identities are encoded by

$$\mathsf{d}_A \mathcal{F} = \sum_{n=0}^{\infty} \frac{1}{n!}\, (-1)^{\frac{1}{2}\, n\, (n-1)}\, \ell_{n+1}(\mathcal{F}, A, \dots, A) \ \in \ V_3 \ , \tag{2.15}$$

which vanishes identically as a consequence of the homotopy relations $\mathcal{J}_n(A, \dots, A) = 0$, for all $n \geq 1$, of the L_∞-algebra.

Given two generalized gauge theories with underlying L_∞-algebras $(V, \{\ell_n\})$ and $(V', \{\ell'_n\})$, an L_∞-morphism $\{\psi_n\}$ between them relates their classical moduli spaces $\mathscr{M}$ and $\mathscr{M}'$ in the following way [17]. A gauge field $A \in V_1$ is sent by an L_∞-morphism $\{\psi_n\}$ into the gauge field

$$A'(A) = \sum_{n=1}^{\infty} \frac{1}{n!}\, (-1)^{\frac{1}{2}\, n\, (n-1)}\, \psi_n(A, \dots, A) \ \in \ V'_1 \ , \tag{2.16}$$

such that the corresponding Euler–Lagrange derivative $\mathcal{F}(A) \in V_2$ is mapped to

$$\mathcal{F}'(A') = \mathcal{F}'(\mathcal{F}, A) = \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^{\frac{1}{2} n (n-1)} \psi_{n+1}(\mathcal{F}, A, \ldots, A) \ \in \ V_2' \ . \tag{2.17}$$

It follows that the Euler–Lagrange locus $\mathcal{F}(A) = 0$ is mapped to the Euler–Lagrange locus $\mathcal{F}'(A') = 0$. An L_∞-morphism also sends gauge orbits into gauge orbits: A gauge variation $\delta_\lambda A$ for $\lambda \in V_0$ is mapped by $\{\psi_n\}$ to the gauge variation $\delta_{\lambda'} A'$ where

$$\lambda'(\lambda, A) = \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^{\frac{1}{2} n (n-1)} \psi_{n+1}(\lambda, A, \ldots, A) \ \in \ V_0' \ . \tag{2.18}$$

It follows that gauge equivalence classes of on-shell solutions $\mathcal{F}(A) = 0$ are sent to gauge equivalence classes of on-shell solutions $\mathcal{F}'(A') = 0$:

$$\begin{aligned} A'(A + \delta_\lambda A) &= A'(A) + \delta'_{\lambda'(\lambda,A)} A'(A) \ , \\ \mathcal{F}'(\mathcal{F} + \delta_\lambda \mathcal{F}, A + \delta_\lambda A) &= \mathcal{F}'(\mathcal{F}, A) + \delta'_{\lambda'(\lambda,A)} \mathcal{F}'(\mathcal{F}, A) \ , \end{aligned} \tag{2.19}$$

with the closure relation (2.14) mapping as

$$A'(A + \delta_{[\![\lambda_1,\lambda_2]\!]_A} A + \Delta_{\lambda_1,\lambda_2} A) = A'(A) + \delta'_{[\![\lambda_1',\lambda_2']\!]'_{A'} + \lambda'(\lambda_2,\delta_{\lambda_1} A) - \lambda'(\lambda_1,\delta_{\lambda_2} A)} A'(A) + \Delta'_{\lambda_1',\lambda_2'} A' \ . \tag{2.20}$$

In particular, if $\{\psi_n\}$ is an L_∞-quasi-isomorphism, then the corresponding moduli spaces $\mathscr{M}$ and $\mathscr{M}'$ of physical states are isomorphic [18,39,40]. However, it is important to note that the converse is not true, and it may happen that two classical field theories have isomorphic moduli spaces while their underlying L_∞-algebras are *not* quasi-isomorphic; we shall see some explicit instances of this later on. The transformations (2.16)–(2.20) are interpreted in terms of Seiberg–Witten maps in [41].

If the symmetries themselves have non-trivial symmetries, that is, there are further gauge redundancies in the description and the gauge symmetries are reducible, then the cochain complex (2.10) may be extended into negative degrees V_{-k} for $k \geq 1$, which are the spaces of "higher gauge transformations", together with their duals V_{k+3}; the space V_{-1} is the vector space of gauge transformations of the gauge parameters, V_{-2} contains gauge variations of the gauge transformations of gauge parameters, and so on. These higher gauge symmetries are encoded as

$$\delta_{(\lambda_{-k-1},A)} \lambda_{-k} = \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^{\frac{1}{2} n (n+1)} \ell_{n+1}(A, \ldots, A, \lambda_{-k-1}) \ \in \ V_{-k} \ , \tag{2.21}$$

where $\lambda_{-k} \in V_{-k}$ for $k \geq 0$. At the classical level in which we are presently working, their inclusion is purely algebraic and only serves to alter the cohomology $H^\bullet(V, \ell_1)$ of the underlying cochain complex at its extremities, leaving the moduli space $\mathscr{M}$ of classical states unchanged.

One way to encode the action functional of the gauge field theory is via a symmetric non-degenerate bilinear pairing $\langle -, - \rangle : V \otimes V \to \mathbb{R}$ of degree -3, as described earlier, which makes V into a cyclic L_∞-algebra. More specifically, the pairing is only defined when restricted to compactly supported sections due to the usual underlying integration. Said otherwise, the pairing is fiber-wise non-degenerate when viewed as a map into the density bundle. The only non-trivial pairings are

$$\langle -, - \rangle : V_1 \otimes V_2 \longrightarrow \mathbb{R} \qquad \text{and} \qquad \langle -, - \rangle : V_0 \otimes V_3 \longrightarrow \mathbb{R} \ ,$$

and we shall explicitly make use of the cyclicity properties

$$\langle A_0, \ell_n(A_1, A_2, \ldots, A_n) \rangle = \langle A_1, \ell_n(A_0, A_2, \ldots, A_n) \rangle \tag{2.22}$$

and

$$\langle \lambda, \ell_{n+1}(E, A_1, \dots, A_n)\rangle = -\langle E, \ell_{n+1}(\lambda, A_1, \dots, A_n)\rangle \tag{2.23}$$

for all $\lambda \in V_0$, $A_0, A_1, \dots, A_n \in V_1$, $E \in V_2$ of suitable compact support and $n \geq 1$. In this case it is easy to see that the field equations $\mathcal{F}(A) = 0$ follow from varying the action functional defined as

$$S(A) := \sum_{n=1}^{\infty} \frac{1}{(n+1)!}\,(-1)^{\frac{1}{2}\,n\,(n-1)}\,\langle A, \ell_n(A, \dots, A)\rangle \ , \tag{2.24}$$

since then cyclicity implies $\delta S(A) = \langle \mathcal{F}, \delta A\rangle$. Cyclicity also implies

$$\delta_\lambda S(A) = \langle \mathcal{F}, \delta_\lambda A\rangle = -\langle \mathsf{d}_A\mathcal{F}, \lambda\rangle \ ,$$

so that gauge invariance of the action functional $\delta_\lambda S(A) = 0$ is then equivalent to the Noether identities $\mathsf{d}_A\mathcal{F}(A) = 0$. Cyclic L_∞-morphisms relate the action functionals of generalized gauge theories.

BV–BRST formulation

The convention we adopt here for the grading of our field theory L_∞-algebra is opposite to the convention of [15]. This bears no significant mathematical effect, except for rendering more direct the duality to the BV–BRST formalism, which makes precise the relation between the L_∞-algebra formulation and the geometric formulation based on the cotangent bundle (2.9); we review this framework in Section 6 for the specific gauge field theories of interest in this paper. In particular, this gives a rigorous description of the quotient defining the moduli space $\mathscr{M}$ of classical solutions, by combining the Koszul–Tate resolution of the quotient by the ideal of Euler–Lagrange derivatives and the Chevalley–Eilenberg resolution of the quotient by gauge transformations. The duality between the L_∞-algebras for classical gauge field theories and their BV–BRST formalism is precisely the duality discussed in Section 2.2, see e.g. [17] for an extensive review; our sign conventions also differ from those of [17] where a sign factor $(-1)^{\frac{1}{2}\,n\,(n-1)}$ is included in the definition of the n-brackets (2.8) as well as in the definition of the L_∞-morphism $\{\psi_n\}$ corresponding to a cohomomorphism Ψ. In this dual formulation, L_∞-quasi-isomorphisms relate physically equivalent generalized gauge field theories at the classical level of moduli spaces and observables, and can also provide useful information in their pertubative quantisation; see e.g. [42] [43]. We will point out instances of this throughout the text. Reducible symmetries now become important as gauge parameters are promoted to dynamical fields in the BV–BRST framework, called "ghosts", and higher gauge parameters become "ghosts-for-ghosts", and so on with the purpose of resolving degeneracies of the corresponding action functional.

Strictly speaking, the L_∞-algebras that arise in field theories are not of finite type over $\mathbb{R}$. However, the underlying vector spaces consist of sections of vector bundles, and the brackets are polydifferential operators of finite degree. Hence the L_∞-algebras are 'local' in the sense of [36]. Thus after factoring the brackets through the appropriate jet bundles, these are all pointwise of finite type over $\mathbb{R}$. This will be enough for the formal dualization that is needed for our purposes in the following.

3 d-dimensional gravity in the Einstein–Cartan–Palatini formalism

In this section we introduce the generalized gauge field theories of interest in this paper. We review the formulation of general relativity in arbitrary dimension d and signature within the Einstein–Cartan formulation, which enables us to treat gravity as a generalized gauge theory on a principal bundle. We shall then review the Palatini action functional for Einstein–Cartan gravity and the role played by Noether identities in this theory.

3.1 Fields

The data of the Einstein–Cartan–Palatini (ECP) formulation of gravity in d dimensions is as follows. The background spacetime consists of a smooth d-dimensional oriented manifold M that admits a pseudo-Riemannian structure of signature (p,q), with

$$d = p + q \ ,$$

where all constructions of this paper hold for any spacetime signature (p,q). Let $\mathscr{V}$ be a vector bundle on M, isomorphic to the tangent bundle TM, which is equipped with a fixed metric η of signature (p,q) and an orientation on its fibers; we shall sometimes refer to $\mathscr{V}$ as the "fake tangent bundle". The field content then consists of two fields which are *a priori* independent: a coframe field and a spin connection.

The coframe field is an orientation-preserving bundle map $e : TM \to \mathscr{V}$, covering the identity, from the tangent bundle TM to the vector bundle $\mathscr{V}$; it can be regarded as a one-form on M valued in $\mathscr{V}$ and used to pull back η to a (possibly degenerate) metric $g = e^*\eta$ on M of indefinite signature (p,q). When the spacetime M is parallelizable, one can take $\mathscr{V} = M \times \mathbb{R}^{p,q}$ to be the trivial bundle and regard e as a globally defined one-form on M with values in $\mathbb{R}^{p,q}$. In that case we write $e = e^a\, \mathtt{E}_a \in \Omega^1(M, \mathbb{R}^{p,q})$, where $e^a \in \Omega^1(M)$ can be expanded as $e^a = e^a_\mu\, \mathrm{d}x^\mu$ in a local holonomic coframe on M. This satisfies[10] $e^a_\mu\, \eta_{ab}\, e^b_\nu = g_{\mu\nu}$, $\mu, \nu = 1, \dots, d$ which gives the components of the (possibly degenerate) dynamical pseudo-Riemannian metric g in any local coordinate chart of M, where η is the standard metric on $\mathbb{R}^{p,q}$ and $\mathtt{E}_a$, $a = 1, \dots, d$ form the canonical oriented pseudo-orthonormal basis of $\mathbb{R}^{p,q}$. When M is not parallelizable the discussion which follows will only be valid on local trivialisations of $\mathscr{V}$. This subtlety is usually ignored in treatments of the first-order formalism for general relativity in the literature, as for physical applications such global issues are usually irrelevant, and in this paper we will also follow this convention for the most part. We shall return to this point in Section 5.3.

The spin connection ω is an $\mathsf{SO}_+(p,q)$-connection on the principal $\mathsf{SO}_+(p,q)$-bundle $\mathscr{P} \to M$ associated to $\mathscr{V}$, so that $\omega \in \Omega^1(\mathscr{P}, \mathfrak{so}(p,q))$; it corresponds to local pseudo-orthogonal transformations of the coframe field that are connected to the identity, which are parameterized by maps from M to the connected component $\mathsf{SO}_+(p,q)$ of the indefinite special orthogonal group $\mathsf{SO}(p,q)$. With the same caveats as discussed above for the coframe fields, we shall usually regard it as a one-form $\omega \in \Omega^1(M, \mathfrak{so}(p,q))$ and write it as[11]

$$\omega = \omega^a{}_b\, \mathtt{E}^b{}_a = \omega^{ab}\, \mathtt{E}_{ba} = \omega^{ab}\, \mathtt{E}_{[ba]}$$

with $\omega^{ab} = -\omega^{ba} \in \Omega^1(M)$, where indices are raised and lowered with the metric η and $\mathtt{E}^b{}_a$ are the $d \times d$ matrix units with matrix elements $(\mathtt{E}^b{}_a)_{cd} = \delta_{ac}\, \delta^b{}_d$.

The covariant derivative of e defines the torsion T of the $\mathsf{SO}_+(p,q)$-connection which can be regarded as a two-form on M valued in $\mathscr{V}$, while the covariant derivative of ω defines its curvature R which can be regarded as a two-form on M with values in the endomorphism bundle of the vector bundle $\mathscr{V}$ with structure group $\mathsf{SO}_+(p,q)$; equivalently, R can be regarded as valued in the second

[10] Here and in the following we always use the Einstein summation convention: repeated upper and lower indices are implicitly summed over.

[11] Here and in the following the square parantheses always mean antisymmetrization over the enclosed indices.

exterior power $\bigwedge^2 \mathscr{V}$ of $\mathscr{V}$, which is isomorphic to the vector bundle $\mathscr{P} \times_{\mathrm{ad}} \mathfrak{so}(p,q)$ associated to $\mathscr{P}$ by the adjoint representation of the structure group. Explicitly,

$$
\begin{aligned}
T &:= \mathrm{d}^{\omega} e = \mathrm{d}e + \omega \wedge e \ \in \ \Omega^2(M, \mathscr{V}) \ , \\
R &:= \mathrm{d}^{\omega} \omega = \mathrm{d}\omega + \tfrac{1}{2}\,[\omega, \omega] \ \in \ \Omega^2\big(M, \mathscr{P} \times_{\mathrm{ad}} \mathfrak{so}(p,q)\big) \ .
\end{aligned}
$$

In a local trivialisation, the wedges here mean matrix multiplication followed by the exterior product on form entries, that is, $\omega \wedge e = \omega^a{}_b \wedge e^b\, \mathtt{E}_a$ and $\frac{1}{2}[\omega, \omega] = \omega^a{}_b \wedge \omega^{bc}\, \mathtt{E}_{[ca]}$. The Bianchi identities are

$$
\mathrm{d}^{\omega} T = R \wedge e \qquad \text{and} \qquad \mathrm{d}^{\omega} R = 0 \ . \tag{3.1}
$$

3.2 Gauge symmetries

The natural internal symmetries of the fields are the finite $\mathsf{SO}_+(p,q)$-transformations $h : M \to \mathscr{P} \times_{\mathrm{Ad}} \mathsf{SO}_+(p,q)$ given by

$$
e \longmapsto h^{-1}\, e \qquad \text{and} \qquad \omega \longmapsto h^{-1}\, \omega\, h + h^{-1}\, \mathrm{d}h \ ,
$$

corresponding to vertical automorphisms of the principal $\mathsf{SO}_+(p,q)$-bundle $\mathscr{P} \to M$ which cover the identity diffeomorphism on M. There is also the action of finite diffeomorphisms $\phi : M \to M$ given by the pullbacks

$$
e \longmapsto \phi^* e \qquad \text{and} \qquad \omega \longmapsto \phi^* \omega \ ,
$$

which map the fields to sections and connections on the corresponding pullback bundles. The infinitesimal version of diffeomorphisms is not compatible with the global structure of the fields (e, ω), as we will discuss further in Section 5.3. For the time being we shall work with the local formulation of the field content (e, ω) of the Einstein–Cartan–Palatini theory and hence consider only the infinitesimal versions of these symmetries; that is, we consider a parallelizable spacetime M and identify the fields as globally defined one-forms $e \in \Omega^1(M, \mathbb{R}^{p,q})$ and $\omega \in \Omega^1(M, \mathfrak{so}(p,q))$.

For any infinitesimal gauge parameter function $\rho : M \to \mathfrak{so}(p,q)$, which corresponds to a local pseudo-orthogonal rotation, we may write $\rho = \rho^{ab}\, \mathtt{E}_{ba}$ for $\rho^{ab} = -\rho^{ba} \in C^\infty(M)$. Then the internal (infinitesimal) gauge transformations are given by

$$
\delta_\rho e = -\rho \cdot e \qquad \text{and} \qquad \delta_\rho \omega = \mathrm{d}^{\omega} \rho := \mathrm{d}\rho + [\omega, \rho] \ . \tag{3.2}
$$

Since e transforms in the fundamental representation of $\mathsf{SO}_+(p,q)$,[12] the notation $\rho \cdot e$ literally means matrix multiplication of a vector: $\rho \cdot e = \rho^a{}_b\, e^b\, \mathtt{E}_a$.

In addition to local pseudo-orthogonal rotations, there is also the standard diffeomorphism gauge symmetry of general relativity. Infinitesimal diffeomorphisms correspond to vector fields on the spacetime M. For an infinitesimal diffeomorphism $\xi \in \Gamma(TM)$, the corresponding gauge transformations are given by the action of the Lie derivative L_ξ on forms:

$$
\delta_\xi e = \mathrm{L}_\xi e \qquad \text{and} \qquad \delta_\xi \omega = \mathrm{L}_\xi \omega \ . \tag{3.3}
$$

The Lie derivatives can be evaluated by using Cartan's 'magic formula'

$$
\mathrm{L}_\xi = \mathrm{d} \circ \iota_\xi + \iota_\xi \circ \mathrm{d} \tag{3.4}
$$

[12] In gauge theory parlance, the coframe field e could be regarded as a matter field, as it is a section of $\mathscr{V} \otimes T^*M$ with $\mathscr{V}$ the vector bundle associated to $\mathscr{P}$ by the fundamental representation, but since e represents a gravitational field we refrain from using this terminology to avoid confusion.

where ι_ξ denotes contraction with the vector field ξ.

Altogether, the involutive symmetry distribution $\mathscr{D}$ on the space of fields

$$\Omega^1(M,\mathbb{R}^{p,q}) \times \Omega^1\big(M,\mathfrak{so}(p,q)\big) \tag{3.5}$$

is the Lie algebra of gauge symmetries generated by the action of the semi-direct product

$$\Gamma(TM) \ltimes \Omega^0\big(M,\mathfrak{so}(p,q)\big) \ . \tag{3.6}$$

Under the infinitesimal symmetries $(\xi,\rho) \in \Gamma(TM) \times \Omega^0\big(M,\mathfrak{so}(p,q)\big)$, the torsion and curvature fields transform as

$$\delta_{(\xi,\rho)}T = \mathrm{L}_\xi T - \rho\cdot T \qquad \text{and} \qquad \delta_{(\xi,\rho)}R = \mathrm{L}_\xi R + [R,\rho] \ .$$

Since $\omega^{ab} = -\omega^{ba}$, under the isomorphism $\mathfrak{so}(p,q) \simeq \bigwedge^2(\mathbb{R}^{p,q})$ given by $\omega^{ab}\,\mathtt{E}_{[ba]} \mapsto \omega^{[ab]}\,\mathtt{E}_a\wedge\mathtt{E}_b$, we will consider the connection as an element $\omega = \omega^{ab}\,\mathtt{E}_a\wedge\mathtt{E}_b \in \Omega^1(M,\bigwedge^2(\mathbb{R}^{p,q}))$ and the curvature as an element $R = R^{ab}\,\mathtt{E}_a\wedge\mathtt{E}_b \in \Omega^2(M,\bigwedge^2(\mathbb{R}^{p,q}))$. This identification has the advantage of making our formulas later on much more compact, avoiding extensive use of indices. This also shows that there is an isomorphism of representations of $\mathsf{SO}_+(p,q)$ given by

$$[\rho,\omega] \longmapsto \rho\cdot(\omega^{ab}\,\mathtt{E}_a\wedge\mathtt{E}_b) = \rho^a{}_c\,\omega^{cb}\,\mathtt{E}_a\wedge\mathtt{E}_b + \rho^b{}_c\,\omega^{ac}\,\mathtt{E}_a\wedge\mathtt{E}_b \ ,$$

where the right-hand side is computed by using the Leibniz rule (or trivial coproduct) for the action of $\mathfrak{so}(p,q)$ on the two-vector representation $\bigwedge^2(\mathbb{R}^{p,q})$. This can be used in comparing the actions of a spin connection on another spin connection in the adjoint representation and on a coframe field in the vector representation to get the useful identity

Lemma 3.7. *If $e_1,\dots,e_{d-2} \in \Omega^1(M,\mathbb{R}^{p,q})$ and $\omega,\omega' \in \Omega^1(M,\mathfrak{so}(p,q))$, then*

$$e_1 \curlywedge \cdots \curlywedge e_{d-2} \curlywedge [\omega,\omega'\,] = \sum_{i=1}^{d-2} e_1 \curlywedge \cdots \widehat{e_i} \cdots \curlywedge e_{d-2} \curlywedge (\omega\wedge e_i) \curlywedge \omega' \tag{3.8}$$

in $d\geq 3$ dimensions, where the $\curlywedge$-product means the exterior products of both the differential form parts and the internal vector space parts,[13] while $\widehat{e_i}$ means omission of the i-th term in the product.

Proof. Using the invariance of a top exterior vector in $\mathbb{R}^{p,q}$ under $\mathsf{SO}_+(p,q)$-transformations yields

$$0 = \omega\wedge(e_1\curlywedge\cdots\curlywedge e_{d-2}\curlywedge\omega'\,) = \omega\wedge(e_1\curlywedge\cdots\curlywedge e_{d-2})\curlywedge\omega' + (-1)^{d-2}e_1\curlywedge\cdots\curlywedge e_{d-2}\curlywedge[\omega,\omega'\,] \ ,$$

and expanding $\omega\wedge(e_1\curlywedge\cdots\curlywedge e_{d-2})$ using the Leibniz rule gives the sum on the right-hand side of (3.8). □

Throughout the text, when considering gravity we shall use the $\curlywedge$-product to denote the double exterior product, while the $\wedge$-product will be reserved for acting via the multivector representation and wedging the respective spacetime forms, unless otherwise explicitly noted.

[13] The $\curlywedge$-product gives $\Omega^\bullet(M,\bigwedge^\bullet\mathbb{R}^{p,q})$ the structure of a graded commutative algebra under the combined form degrees. It does not associate with the $\wedge$-product acting via multi-vector representations, as will be apparent in calculations below.

3.3 Field equations

Given the field content from Section 3.1, the action functional for Einstein–Cartan–Palatini gravity in $d > 2$ dimensions is then given by

$$S_{\rm ECP}(e,\omega) := \frac{1}{2\,\kappa^2}\int_M \mathrm{Tr}\Big(\frac{1}{d-2}\,e^{d-2}\curlywedge R + \frac{1}{d}\,\Lambda\,e^d\Big)\ , \tag{3.9}$$

where the square of the parameter $\kappa \in \mathbb{R}$ gives the gravitational constant and $\Lambda \in \mathbb{R}$ is the cosmological constant. The integrands in (3.9) are d-forms on M with values in $\bigwedge^d \mathscr{V}$, and since $\mathscr{V}$ has structure group $\mathsf{SO}_+(p,q)$ it carries a natural volume form such that the Hodge duality operator $\mathrm{Tr} : \Omega^d(M,\bigwedge^d\mathscr{V}) \to \Omega^d(M)$ extracts the canonical choice of function defined by a section of $\bigwedge^d\mathscr{V}$. The powers are taken with respect to the $\curlywedge$-product, so that $e^{d-2}\curlywedge R$ and e^d are elements in $\Omega^d(M,\bigwedge^d\mathscr{V})$. In a local trivialization, the Hodge duality operator $\mathrm{Tr} : \bigwedge^d(\mathbb{R}^{p,q}) \to \mathbb{R}$ on the internal vector space has the normalization

$$\mathrm{Tr}(\mathtt{E}_{a_1}\wedge\cdots\wedge\mathtt{E}_{a_d}) = \varepsilon_{a_1\cdots a_d}\ ,$$

where $\varepsilon_{a_1\cdots a_d}$ is the Levi–Civita symbol in d dimensions, thus yielding an $\mathbb{R}$-valued d-form on the d-dimensional manifold M, which we can integrate. The action functional (3.9) is invariant under both finite and infinitesimal gauge symmetries of Section 3.2. For $d > 3$ one can also add to (3.9) higher curvature terms $e^{d-2\,k}\curlywedge R^k$ which give Lovelock theories of gravity, but we will shall not consider such extensions in this paper.

The field equations which follow from varying the action functional (3.9) with respect to compactly supported (e,ω) are given by

$$\begin{aligned}\mathcal{F}_e(e,\omega) &:= e^{d-3}\curlywedge R + \Lambda\,e^{d-1} = 0 \ \in\ \Omega^{d-1}\big(M,\textstyle\bigwedge^{d-1}\mathscr{V}\big)\ ,\\ \mathcal{F}_\omega(e,\omega) &:= e^{d-3}\curlywedge T = 0 \ \in\ \Omega^{d-1}\big(M,\textstyle\bigwedge^{d-2}\mathscr{V}\big)\ ,\end{aligned} \tag{3.10}$$

where the first equation comes from varying with respect to e and the second equation with respect to ω using the identity (3.8). In the case of a gravitational field that has no singularities, where the coframe field e is everywhere invertible and so defines a non-degenerate metric $g = \eta_{ab}\,e^a\otimes e^b$, this formulation is equivalent on-shell to the metric formulation with the Einstein–Hilbert action functional (1.1): In this case the second equation is equivalent to $T = 0$, which is just the condition that the $\mathsf{SO}_+(p,q)$-connection is torsion-free. This can be uniquely solved (up to gauge equivalence) to give ω in terms of e, which may then be identified with the Levi–Civita connection for the metric g; metric compatibility follows from the fact that ω is an $\mathsf{SO}_+(p,q)$-connection. The first equation is then equivalent to the usual vacuum Einstein field equations with cosmological constant.

Thus the Einstein–Cartan–Palatani field theory, including degenerate coframes, only recovers the standard Einstein–Hilbert metric formulation of general relativity on-shell as a particular subspace of its Euler–Lagrange locus. The two theories are generally not equivalent, not even on-shell. However, it is important in our case to allow for degenerate coframes, so that the space of fields (3.5) is indeed a vector space, which is a necessary requirement for the L_∞-algebra formulation. This is not merely a technical burden, since the resulting module structures of the space of fields and that of the L_∞-algebras'[14] render the theory very amenable to twisting methods [28]. Furthermore, it allows for the study of invertible morphisms up to homotopy in the L_∞-algebras category which have direct physical content. This is in marked contrast to the Einstein–Hilbert formulation, which necessitates a restriction to non-degenerate metrics in order to write down the action functional (1.1); however, the space of non-degenerate metrics is not a vector space. In the remainder of this paper, the gravitational constant will not play any role and we shall normalize the action functional (3.9) so that $2\,\kappa^2 = 1$.

[14]The structures turn out to be modules of the enveloping algebra of vector fields on M.

3.4 Noether identities

Noether's first theorem, which asserts the existence of a weakly conserved current for each global symmetry of an action functional, is not relevant in the present context because the action functional for general relativity does not have global symmetries.[15] On the other hand, Noether's second theorem, which relates the Euler–Lagrange derivatives of an action functional off-shell, applies to the local pseudo-orthogonal and diffeomorphism gauge symmetries of the Einstein–Cartan–Palatini theory, see e.g. [44, 45]. The corresponding Noether identities follow from gauge-invariance of the action functional (3.9):

$$0 = \delta_{(\xi,\rho)} S_{\rm ECP}(e,\omega) = \int_M \mathrm{Tr}\Big(\mathcal{F}_e(e,\omega) \curlywedge \delta_{(\xi,\rho)}e + \mathcal{F}_\omega(e,\omega) \curlywedge \delta_{(\xi,\rho)}\omega\Big) \ ,$$

where $\delta_{(\xi,\rho)}(e,\omega) := (\delta_\xi e+\delta_\rho e, \delta_\xi\omega+\delta_\rho\omega)$, and then varying this equation with respect to compactly supported (ξ,ρ) using the explicit expressions for the gauge transformations from (3.2) and (3.3). This leads to a pair of strong differential identities among the Euler–Lagrange derivatives:

$$\mathsf{d}_{(e,\omega)}(\mathcal{F}_e,\mathcal{F}_\omega) = (0,0) \ \in \ \Omega^1\big(M,\Omega^d(M)\big) \times \Omega^d\big(M,\textstyle\bigwedge^{d-2}(\mathbb{R}^{p,q})\big) \ , \tag{3.11}$$

where

$$\begin{aligned}\mathsf{d}_{(e,\omega)}(\mathcal{F}_e,\mathcal{F}_\omega) := \Big(&- \mathrm{d}x^\mu \otimes \mathrm{Tr}\big(\iota_\mu \mathrm{d}e \curlywedge \mathcal{F}_e + (-1)^{d-1}\, \iota_\mu \mathrm{d}\omega \curlywedge \mathcal{F}_\omega - \iota_\mu e \curlywedge \mathrm{d}\mathcal{F}_e \\ &- (-1)^{d-1}\, \iota_\mu\omega \curlywedge \mathrm{d}\mathcal{F}_\omega\big) \ , \ -\frac{d-1}{2}\,\mathcal{F}_e \wedge e + (-1)^{d-1}\, \mathrm{d}^\omega \mathcal{F}_\omega\Big) \ , \end{aligned} \tag{3.12}$$

and ι_μ denotes the contraction with vectors $\partial_\mu = \frac{\partial}{\partial x^\mu}$ of the local holonomic frame dual to the basis $\{\mathrm{d}x^\mu\}$ of one-forms in a local coordinate chart on M. Here we identify the vector space of one-forms valued in d-forms $\Omega^1\big(M,\Omega^d(M)\big)$ with $\Omega^1(M)\otimes\Omega^d(M)$, and

$$\mathcal{F}_e \wedge e := \big(\mathcal{F}_e^{a_1\cdots a_{d-2}\,k} \wedge \eta_{kl}\, e^l\big)\, \mathtt{E}_{a_1} \wedge \cdots \wedge \mathtt{E}_{a_{d-2}} \ ,$$

that is, one uses the flat metric to identify e with an element of $\Omega^1(M,(\mathbb{R}^{p,q})^*)$ and then contracts with the multivector part in $\bigwedge^{d-1}(\mathbb{R}^{p,q})$, and takes the exterior product of the differential forms.[16]

The converse of Noether's second theorem can be used to work backwards from this identity to deduce that the action functional (3.9) has local pseudo-orthogonal and diffeomorphism gauge symmetries [45]. The first component is the Noether identity corresponding to local diffeomorphism invariance $\delta_\xi S_{\rm ECP}=0$. The second component gives the Noether identity corresponding to the local pseudo-orthogonal gauge symmetry $\delta_\rho S_{\rm ECP}=0$, which also follows from the first Bianchi identity in (3.1) by taking the covariant derivative of the second Euler–Lagrange derivative in (3.10):

$$\begin{aligned}\mathrm{d}^\omega \mathcal{F}_\omega &= (d-3)\, \mathrm{d}^\omega e \curlywedge e^{d-4} \curlywedge T + (-1)^{d-3}\, e^{d-3} \curlywedge \mathrm{d}^\omega T \\ &= (d-3)\, T \curlywedge e^{d-4} \curlywedge T + (-1)^{d-3}\, e^{d-3} \curlywedge R \wedge e \\ &= (-1)^{d-3}\, \frac{d-1}{2}\, \big(\mathcal{F}_e - \Lambda\, e^{d-1}\big) \wedge e \\ &= (-1)^{d-3}\, \frac{d-1}{2}\, \mathcal{F}_e \wedge e \ ,\end{aligned}$$

where in the third equality we used $T\curlywedge T=0$ and the first Euler–Lagrange derivative from (3.10), while in the last equality we used $(e\curlywedge e)\wedge e = (e^a \wedge \eta_{bc}\, e^b \wedge e^c)\, \mathtt{E}_a = 0$. The overall prefactor appears due to our convenient conventions on $\bigwedge^\bullet(\mathbb{R}^{p,q})$, whereby the contraction in question may be checked to act as a derivation with respect to the exterior product up to the overall factors. For further discussion of Noether's second theorem in the first order formalism for gravity, see [44].

[15] Of course, specific solutions may have symmetries generated by Killing vectors that can be used to produce conserved quantities, but there are no global symmetries *a priori*.

[16] For $d=3$ this reduces to the action of $\mathfrak{so}(p,q)\simeq\bigwedge^2(\mathbb{R}^{p,q})$ on $\mathbb{R}^{p,q}$.

4 Topological gauge theories with redundant symmetries

The ECP action functional (3.9) bears a striking similarity with the action functionals of a class of topological field theories known as BF theories. This observation has been the source of intense investigations into the formulation of gravity as a deformation of BF theories, and particularly in certain approaches to quantum gravity (see e.g. [33] for a review). We will come back to these connections in Sections 7 and 8. Before coming to the L_∞-algebra formulation of the ECP gravity theory from Section 3, it is therefore a useful warm-up to look at some simpler examples of Schwarz-type topological gauge theories[17] whose dynamical L_∞-algebras are straightfoward to formulate, yet they involve many features of the more complicated L_∞-algebras that we look at in later sections. They also serve to illustrate some important points concerning the roles of diffeomorphisms, redundant symmetries, and classical equivalences between field theories, that will be particularly pertinent to the discussions in Sections 7 and 8.

4.1 Chern–Simons theory in the L_∞-algebra formalism

Chern–Simons theory in three spacetime dimensions provides the basic example of an L_∞-algebra in gauge theory with a very natural and simple bracket structure, see e.g. [15, 17]. For later use, we shall spell out the details and use them to illustrate how redundant symmetries of a field theory are naturally handled by the L_∞-algebra framework.

Let G be a Lie group whose Lie algebra $\mathfrak{g}$ is endowed with an invariant quadratic form $\mathrm{Tr}_\mathfrak{g} : \mathfrak{g}\otimes\mathfrak{g} \to \mathbb{R}$; invariance means $\mathrm{Tr}_\mathfrak{g}([X,Y]_\mathfrak{g} \otimes Z) = \mathrm{Tr}_\mathfrak{g}(X \otimes [Y,Z]_\mathfrak{g})$ for all $X,Y,Z \in \mathfrak{g}$, where $[-,-]_\mathfrak{g}$ is the Lie bracket in $\mathfrak{g}$. Let $\mathscr{P} \to M$ be a principal G-bundle on an oriented three-dimensional manifold M, which for simplicity we assume to be trivial, $\mathscr{P} = M \times \mathsf{G}$, so that its connections can be regarded as one-forms on M with values in $\mathfrak{g}$; this restriction will be enough for our purposes later on. The Lie bracket on $\mathfrak{g}$ is extended to $\Omega^\bullet(M,\mathfrak{g}) := \Omega^\bullet(M)\otimes\mathfrak{g}$ by

$$[\alpha\otimes X, \beta\otimes Y]_\mathfrak{g} := \alpha\wedge\beta\otimes[X,Y]_\mathfrak{g} \ .$$

The Chern–Simons action functional for a gauge field $A \in \Omega^1(M,\mathfrak{g})$ is then defined by [18]

$$S_{\rm CS}(A) := \frac{1}{2}\int_M \mathrm{Tr}_\mathfrak{g}\Big(A\wedge\mathrm{d}A + \frac{1}{3}\,A\wedge[A,A]_\mathfrak{g}\Big) \ . \tag{4.1}$$

This action functional is invariant under the gauge transformations

$$\delta_\lambda A = \mathrm{d}^A\lambda := \mathrm{d}\lambda + [A,\lambda]_\mathfrak{g} \ , \tag{4.2}$$

where $\lambda \in \Omega^0(M,\mathfrak{g})$. The Chern–Simons field equations state that the G-connection A is flat, that is, its curvature vanishes:

$$\mathcal{F}(A) := F = \mathrm{d}^A A = \mathrm{d}A + \tfrac{1}{2}\,[A,A]_\mathfrak{g} = 0 \ \in \ \Omega^2(M,\mathfrak{g}) \ , \tag{4.3}$$

while the Noether identities are equivalent to the Bianchi identity[19]

$$\mathsf{d}_A\mathcal{F} := \mathrm{d}^A F = \mathrm{d}F + [A,F]_\mathfrak{g} = 0 \ \in \ \Omega^3(M,\mathfrak{g}) \ . \tag{4.4}$$

[17] See e.g. [46] for a review.

[18] Here and in BF theory the wedge symbol within the Trace pairings denotes simply wedging the spacetime form parts, in contrast to the ECP convention.

[19] Note that this is in contrast to what happens in Einstein–Cartan–Palatini theory, where the first Bianchi identity in (3.1) implies the Noether identity for local pseudo-orthogonal rotations, but is not generally equivalent to it.

The classical moduli space $\mathscr{M}_{\rm CS}$ of physical states of Chern–Simons theory is thus the moduli space of flat G-connections on the three-manifold M.

The cochain complex underlying the L_∞-algebra of Chern–Simons gauge theory is simply the de Rham complex in three dimensions with values in the Lie algebra $\mathfrak{g}$:

$$0 \longrightarrow \Omega^0(M,\mathfrak{g}) \xrightarrow{\ \rm d\ } \Omega^1(M,\mathfrak{g}) \xrightarrow{\ \rm d\ } \Omega^2(M,\mathfrak{g}) \xrightarrow{\ \rm d\ } \Omega^3(M,\mathfrak{g}) \xrightarrow{\ \rm d\ } 0 \ .$$

The corresponding graded vector space

$$V^{\rm CS} = \Omega^\bullet(M,\mathfrak{g}) = \Omega^0(M,\mathfrak{g}) \ \oplus \ \Omega^1(M,\mathfrak{g}) \ \oplus \ \Omega^2(M,\mathfrak{g}) \ \oplus \ \Omega^3(M,\mathfrak{g})$$

has non-zero graded components $V_k^{\rm CS} = \Omega^k(M,\mathfrak{g})$ for $k=0,1,2,3$, whose elements we denote respectively by λ, A, $\mathcal{A}$ and $\mathit{\Lambda}$. This yields a four-term L_∞-algebra with 1-bracket defined by the exterior derivative as

$$\ell_1^{\rm CS}(\lambda) = {\rm d}\lambda \ \in \ V_1^{\rm CS} \ , \quad \ell_1^{\rm CS}(A) = {\rm d}A \ \in \ V_2^{\rm CS} \qquad \text{and} \qquad \ell_1^{\rm CS}(\mathcal{A}) = {\rm d}\mathcal{A} \ \in \ V_3^{\rm CS} \ .$$

The 2-brackets are given by the Lie bracket of $\mathfrak{g}$ as

$$\begin{aligned}
\ell_2^{\rm CS}(\lambda_1,\lambda_2) &= -[\lambda_1,\lambda_2]_\mathfrak{g} \ \in \ V_0^{\rm CS} \ , \\
\ell_2^{\rm CS}(\lambda,A) &= -[\lambda,A]_\mathfrak{g} \ \in \ V_1^{\rm CS} \ , \\
\ell_2^{\rm CS}(\lambda,\mathcal{A}) &= -[\lambda,\mathcal{A}]_\mathfrak{g} \ \in \ V_2^{\rm CS} \ , \\
\ell_2^{\rm CS}(\lambda,\mathit{\Lambda}) &= -[\lambda,\mathit{\Lambda}]_\mathfrak{g} \ \in \ V_3^{\rm CS} \ , \\
\ell_2^{\rm CS}(A_1,A_2) &= -[A_1,A_2]_\mathfrak{g} \ \in \ V_2^{\rm CS} \ , \\
\ell_2^{\rm CS}(A,\mathcal{A}) &= -[A,\mathcal{A}]_\mathfrak{g} \ \in \ V_3^{\rm CS} \ ,
\end{aligned}$$

with all other brackets vanishing. Thus the gauge theory is organised by a differential graded Lie algebra: The homotopy relations in this case easily follow from the nilpotence and Leibniz rule for the exterior derivative d, together with the Jacobi identity for the Lie bracket of $\mathfrak{g}$. One easily verifies the kinematical and dynamical structure of Chern–Simons gauge theory from these brackets as designed by the prescription of Section 2.3; in particular the gauge transformations, field equations and Noether identities are encoded as

$$\begin{aligned}
\delta_\lambda A &= \ell_1^{\rm CS}(\lambda) + \ell_2^{\rm CS}(\lambda,A) \ , \\
F &= \ell_1^{\rm CS}(A) - \frac{1}{2}\,\ell_2^{\rm CS}(A,A) \ , \\
\delta_\lambda F &= \ell_2^{\rm CS}(\lambda,F) \ , \\
{\rm d}^A F &= \ell_1^{\rm CS}(F) - \ell_2^{\rm CS}(A,F) \ .
\end{aligned}$$

This can be made into a cyclic L_∞-algebra by defining the pairing of $\mathfrak{g}$-valued forms in complementary degrees:

$$\langle\alpha,\beta\rangle^{\rm CS} := \int_M \, {\rm Tr}_\mathfrak{g}(\alpha\wedge\beta) \ , \tag{4.5}$$

for $\alpha\in\Omega^p(M,\mathfrak{g})$ and $\beta\in\Omega^{3-p}(M,\mathfrak{g})$ with $p=0,1,2,3$. This defines a cyclic non-degenerate pairing (of degree -3) on $V_1^{\rm CS}\otimes V_2^{\rm CS}\to\mathbb{R}$ and $V_0^{\rm CS}\otimes V_3^{\rm CS}\to\mathbb{R}$, where cyclicity follows from the invariance of the quadratic form on the Lie algebra $\mathfrak{g}$. The adjoint of the exterior derivative

$\mathrm{d} : \Omega^p(M,\mathfrak{g}) \to \Omega^{p+1}(M,\mathfrak{g})$ with respect to this inner product is $(-1)^{p+1}\,\mathrm{d}$, which yields the appropriate cyclicity structure from which the Chern–Simons action functional (4.1) is reproduced according to the general prescription of Section 2.3:

$$S_{\rm CS}(A) = \frac{1}{2}\,\langle A, \ell_1^{\rm CS}(A)\rangle^{\rm CS} - \frac{1}{3!}\,\langle A, \ell_2^{\rm CS}(A,A)\rangle^{\rm CS}\ .$$

In this sense, the L_∞-algebra formulation of field theories is a generalization of Chern–Simons theory, which is dual to the BV formalism for Lagrangian field theories; see [17] for further details on this perspective.

4.2 Diffeomorphisms as redundant symmetries

Chern–Simons theory is also the prototypical example of a topological field theory: Its field equation simply state that the G-connection A is flat. In addition, the theory is background independent: There is no background structure assumed on the manifold M, aside from its orientation. Thus the action functional is constructed solely of differential forms built from the dynamical field, and hence it is automatically invariant under orientation-preserving diffeomorphisms of M.[20] In particular, the action of any infinitesimal diffeomorphism $\xi \in \Gamma(TM)$ on a connection A is via the Lie derivative:

$$\delta_\xi A := \mathrm{L}_\xi A\ , \tag{4.6}$$

which leaves the Chern–Simons action functional (4.1) invariant:

$$\delta_\xi S_{\rm CS} = 0\ .$$

The corresponding Noether identity may then be read off by using the Cartan formula (3.4) and integrating by parts to get the strong differential identity

$$\mathrm{d}x^\mu \otimes \mathrm{Tr}_{\mathfrak{g}}(\iota_\mu \mathrm{d}A \wedge F) - \mathrm{d}x^\mu \otimes \mathrm{Tr}_{\mathfrak{g}}(\iota_\mu A \otimes \mathrm{d}F) = 0 \ \in\ \Omega^1\big(M, \Omega^3(M)\big)\ .$$

One may wonder how to reconcile this apparently "extra" symmetry with the usual gauge symmetry of Chern–Simons theory, and whether this can be incorporated in the L_∞-algebra framework. Of course, the answer to the first question is well-known to experts. The crucial point is that the action of any vector field $\xi \in \Gamma(TM)$ may be compensated by the action of a specially chosen gauge transformation $\lambda_\xi \in \Omega^0(M,\mathfrak{g})$, up to a term proportional to the field equations. To see this, note that the variation (4.6) can be written as

$$\begin{aligned}
\delta_\xi A &= \mathrm{d}\iota_\xi A + \iota_\xi \mathrm{d}A \\
&= \mathrm{d}\iota_\xi A + \iota_\xi \mathrm{d}A + \tfrac{1}{2}\,\iota_\xi[A,A]_{\mathfrak{g}} - \tfrac{1}{2}\,\iota_\xi[A,A]_{\mathfrak{g}} \\
&= \mathrm{d}\iota_\xi A + [A, \iota_\xi A]_{\mathfrak{g}} + \iota_\xi F \\
&=: \mathrm{d}^A \lambda_\xi + \iota_\xi \mathcal{F}(A)
\end{aligned}$$

where we defined the "field dependent" gauge transformation $\lambda_\xi := \iota_\xi A \in \Omega^0(M,\mathfrak{g})$. Thus the gauge orbits of the $\Gamma(TM)$-action are included in the gauge orbits of the standard gauge transformations, on-shell. In other words, the traditional moduli space of physical states for Chern–Simons theory $\mathscr{M}_{\rm CS}$ is unchanged if one further quotients by the action of (infinitesimal) diffeomorphisms.[21] One

[20] Orientation-*reversing* diffeomorphisms change the sign of the Chern–Simons action functional.

[21] This statement holds at the infinitesimal level, but fails for finite diffeomorphisms which are not connected to the identity.

then declares the extra symmetries to be *redundant* – they do not extend the distribution $\mathscr{D}$ of gauge transformations on the space of fields.

We will now demonstrate that the L_∞-algebra framework actually serves as a useful tool for encoding redundant symmetries and their relation to the smaller Lie algebra generating the distribution $\mathscr{D}$ of symmetries of the corresponding generalized gauge theory. Indeed, the vector space $V^{\rm CS}$ underlying the L_∞-algebra of Chern–Simons theory may be extended as

$$V^{\rm ext}_{\rm CS} := \Gamma(TM)\times\Omega^0(M,\mathfrak{g}) \ \oplus\ \Omega^1(M,\mathfrak{g}) \ \oplus\ \Omega^2(M,\mathfrak{g}) \ \oplus\ \Omega^1\big(M,\Omega^3(M)\big)\times\Omega^3(M,\mathfrak{g}) \ . \tag{4.7}$$

That is, we simply extend the space of gauge transformations $V_0^{\rm CS}$ to include the extra diffeomorphism gauge symmetry, with elements $\xi\in\Gamma(TM)$, while also extending the space $V_3^{\rm CS}$ to accommodate for the corresponding Noether identity, with elements $\mathcal{X}\in\Omega^1\big(M,\Omega^3(M)\big)$. The brackets are then modified to accommodate for the action of diffeomorphisms on the various spaces: The 1-bracket is modified trivially as

$$\ell_1^{\rm ext}(\xi,\lambda)={\rm d}\lambda \ , \quad \ell_1^{\rm ext}(A)={\rm d}A \qquad \text{and} \qquad \ell_1^{\rm ext}(\mathcal{A})=(0,{\rm d}\mathcal{A}) \ , \tag{4.8}$$

while the modified 2-brackets are given by

$$\begin{aligned}
\ell_2^{\rm ext}\big((\xi_1,\lambda_1)\,,\,(\xi_2,\lambda_2)\big) &= \big([\xi_1,\xi_2]\,,\,{\rm L}_{\xi_1}\lambda_2-{\rm L}_{\xi_2}\lambda_1-[\lambda_1,\lambda_2]_\mathfrak{g}\big) \ , \\
\ell_2^{\rm ext}\big((\xi,\lambda)\,,\,A\big) &= {\rm L}_\xi A-[\lambda,A]_\mathfrak{g} \ , \\
\ell_2^{\rm ext}\big((\xi,\lambda)\,,\,\mathcal{A}\big) &= {\rm L}_\xi \mathcal{A}-[\lambda,\mathcal{A}]_\mathfrak{g} \ , \\
\ell_2^{\rm ext}\big((\xi,\lambda)\,,\,(\mathcal{X},\Lambda)\big) &= \big({\rm d}x^\mu\otimes{\rm Tr}_\mathfrak{g}(\iota_\mu{\rm d}\lambda\otimes\Lambda)+{\rm L}_\xi\mathcal{X}\,,\,-[\lambda,\Lambda]_\mathfrak{g}+{\rm L}_\xi\Lambda\big) \ , \\
\ell_2^{\rm ext}(A_1,A_2) &= -[A_2,A_1]_\mathfrak{g} \ , \\
\ell_2^{\rm ext}(A,\mathcal{A}) &= \big({\rm d}x^\mu\otimes{\rm Tr}_\mathfrak{g}(\iota_\mu{\rm d}A\wedge\mathcal{A})-{\rm d}x^\mu\otimes{\rm Tr}_\mathfrak{g}(\iota_\mu A\otimes{\rm d}\mathcal{A})\,,\,-[A,\mathcal{A}]_\mathfrak{g}\big) \ .
\end{aligned} \tag{4.9}$$

In particular, the first 2-bracket is the Lie bracket for the extended semi-direct product gauge algebra $\Gamma(TM)\ltimes\Omega^0(M,\mathfrak{g})$. The proof of the homotopy relations for these brackets is formally identical to the proof we present for the brackets of three-dimensional gravity in Appendix A.1. These brackets encode all the dynamical data of the gauge theory as prescribed in Section 2.3, now including the action of diffeomorphisms and the corresponding Noether identity.

Since $V_1^{\rm ext}=V_1^{\rm CS}$ and $V_2^{\rm ext}=V_2^{\rm CS}$, the cyclic pairing $\langle-,-\rangle^{\rm ext}$ is given by (4.5) on $V_1^{\rm ext}\otimes V_2^{\rm ext}$. It is further extended on $V_0^{\rm ext}\otimes V_3^{\rm ext}$ by defining

$$\langle(\xi,\lambda)\,,\,(\mathcal{X},\Lambda)\rangle^{\rm ext} := \int_M\,\iota_\xi\mathcal{X}+\int_M\,{\rm Tr}_\mathfrak{g}(\lambda\otimes\Lambda) \ .$$

The extended brackets may then be easily checked to be cyclic with respect to this pairing as well; the calculation in question is carried out in the case of ECP gravity later on.

4.3 L_∞-quasi-isomorphism

Although the moduli spaces of classical solutions in the two L_∞-algebras from Sections 4.1 and 4.2 are the same, the cohomologies of the cochain complexes generated by the differentials $\ell_1^{\rm CS}$ and $\ell_1^{\rm ext}$ are not isomorphic; this is immediately apparent by looking at the cohomology in degree 0 for the two L_∞-algebras:

$$H^0\big(V^{\rm CS},\ell_1^{\rm CS}\big)=H^0(M,\mathfrak{g}) \qquad \text{and} \qquad H^0\big(V^{\rm ext}_{\rm CS},\ell_1^{\rm ext}\big)=\Gamma(TM)\times H^0(M,\mathfrak{g}) \ .$$

This means that there cannot exist an L_∞-quasi-isomorphism between the two L_∞-algebras. However, there is no contradiction here: While two field theories with quasi-isomorphic L_∞-algebras have isomorphic classical moduli spaces, the converse need not necessarily hold.

We can nevertheless describe the classical equivalence between the two L_∞-algebra formulations of Chern–Simons gauge theory as an L_∞-quasi-isomorphism in the following way. The redundancy may be encoded in the L_∞-algebra framework by further extending the complex defined by the vector space $V_{\rm CS}^{\rm ext}$. This is done by introducing a copy of the redundant subspace in degree -1:

$$V_{-1}^{\rm ext} := \Gamma(TM) \ .$$

As with the rest of the complex, this should be supplemented with its dual in degree 4:

$$V_4^{\rm ext} := \Omega^1\big(M, \Omega^3(M)\big) \ .$$

We denote elements of $V_{-1}^{\rm ext}$ by $\check\xi$ and elements of $V_4^{\rm ext}$ by $\check{\mathcal{X}}$ to distinguish them from their copies in degrees 0 and 3, respectively. The issue with the cohomology of $\ell_1^{\rm ext}$ is then fixed by extending it as the inclusion $\ell_1^{\rm ext} : V_{-1}^{\rm ext} \to V_0^{\rm ext}$ and the projection $\ell_1^{\rm ext} : V_3^{\rm ext} \to V_4^{\rm ext}$:

$$\ell_1^{\rm ext}\big(\check\xi\,\big) := \big(\check\xi, 0\big) \ \in \ V_0^{\rm ext} \qquad \text{and} \qquad \ell_1^{\rm ext}(\mathcal{X}, \mathit{\Lambda}) := \mathcal{X} \ \in \ V_4^{\rm ext} \ .$$

The differential condition $\ell_1^{\rm ext} \circ \ell_1^{\rm ext} = 0$ is still satisfied on the extended complex, but now the cohomologies agree as expected: $H^\bullet(V^{\rm CS}, \ell_1^{\rm CS}) = H^\bullet(V_{\rm CS}^{\rm ext}, \ell_1^{\rm ext})$.

To complete the L_∞-algebra extension, one should extend the 2-bracket $\ell_2^{\rm ext}$ while still satisfying the remaining homotopy relations. It is easy to see that the following definition does the job:

$$\begin{aligned}
\ell_2^{\rm ext}\big(\check\xi\,,\,(\xi,\lambda)\big) &:= \big[\check\xi,\xi\big] \ \in \ V_{-1}^{\rm ext} \ , \\
\ell_2^{\rm ext}\big(\check\xi, A\big) &:= \big(0, \iota_{\check\xi} A\big) \ \in \ V_0^{\rm ext} \ , \\
\ell_2^{\rm ext}\big(\check\xi, \mathcal{A}\big) &:= \iota_{\check\xi}\mathcal{A} \ \in \ V_1^{\rm ext} \ , \\
\ell_2^{\rm ext}\big(\check\xi\,,\,(\mathcal{X},\mathit{\Lambda})\big) &:= \iota_{\check\xi}\mathit{\Lambda} \ \in \ V_2^{\rm ext} \ , \\
\ell_2^{\rm ext}\big(\check\xi, \check{\mathcal{X}}\big) &:= \big({\rm L}_{\check\xi}\check{\mathcal{X}}, 0\big) \ \in \ V_3^{\rm ext} \ , \\
\ell_2^{\rm ext}\big((\xi,\lambda)\,,\,\check{\mathcal{X}}\big) &:= {\rm L}_{\xi}\check{\mathcal{X}} \ \in \ V_4^{\rm ext} \ , \\
\ell_2^{\rm ext}\big(A\,,\,(\mathcal{X},\mathit{\Lambda})\big) &:= {\rm d}x^\mu \otimes {\rm Tr}_{\mathfrak{g}}(\iota_\mu A \otimes \mathit{\Lambda}) \ \in \ V_4^{\rm ext} \ , \\
\ell_2^{\rm ext}(\mathcal{A}_1, \mathcal{A}_2) &:= {\rm d}x^\mu \otimes {\rm Tr}_{\mathfrak{g}}(\iota_\mu \mathcal{A}_1 \wedge \mathcal{A}_2) \ \in \ V_4^{\rm ext} \ ,
\end{aligned}$$

where also $\ell_2^{\rm ext}\big(\check\xi_1, \check\xi_2\big) := 0$ as this lands in $V_{-2}^{\rm ext} = 0$. The pairing is further extended to $V_{-1}^{\rm ext} \otimes V_4^{\rm ext}$ by

$$\big\langle \check\xi, \check{\mathcal{X}} \big\rangle^{\rm ext} := \int_M \iota_{\check\xi}\check{\mathcal{X}} \ ,$$

and the new brackets are cyclic with respect to this pairing as well. The proofs of the homotopy relations in this case follow exactly as for the calculations we spell out for gravity in Appendix A.1, with the only non-trivial (but straightforward) check occuring in the graded Jacobi identity on a pair of gauge parameters (ξ, λ), and two Euler–Lagrange derivatives $\mathcal{A}_1$ and $\mathcal{A}_2$.

The redundancy of the space of gauge transformations $V_0^{\rm ext} = \Gamma(TM) \times \Omega^0(M, \mathfrak{g})$ should really be regarded in terms of the "higher gauge transformations" (2.21) which act on it by the redundant symmetries in $V_{-1}^{\rm ext}$: Any $(\xi, \lambda) \in V_0^{\rm ext}$ generates the gauge transformation

$$\delta_{(\xi,\lambda)}^{\rm ext} A = {\rm L}_\xi A + {\rm d}^A \lambda \ ,$$

which is on-shell equivalent to the gauge transformation generated by

$$(\xi', \lambda') := (\xi, \lambda) + \delta^{\rm ext}_{(\check\xi, A)}(\xi, \lambda)$$

for any $\check\xi \in V^{\rm ext}_{-1}$ with

$$\delta^{\rm ext}_{(\check\xi, A)}(\xi, \lambda) := \ell_1^{\rm ext}\big(\check\xi\,\big) - \ell_2^{\rm ext}\big(A, \check\xi\,\big) = \big(\check\xi, -\iota_{\check\xi} A\big) \ \in \ V_0^{\rm ext} \ .$$

Given this equivalence, all vector fields are redundant in the following sense: For any $(\xi, \lambda) \in V_0^{\rm ext}$, pick $\check\xi := -\xi \in V^{\rm ext}_{-1}$. Then

$$\delta^{\rm ext}_{(\xi', \lambda')} A = \delta^{\rm ext}_{(0, \lambda + \iota_\xi A)} A$$

which is equivalent to $\delta^{\rm ext}_{(\xi, \lambda)} A$ up to terms involving Euler–Lagrange derivatives.

This on-shell equivalence is made precise in terms of an *off-shell* L_∞-quasi-isomorphism in the following way. One can eliminate the redundant gauge symmetries by using a quasi-isomorphism $\{\psi_n^{\rm CS}\}$ from the standard Chern–Simons L_∞-algebra of Section 4.1 to the extended L_∞-algebra here, with the component multilinear graded antisymmetric maps

$$\psi_n^{\rm CS} : {\textstyle\bigwedge}^n V^{\rm CS} \longrightarrow V^{\rm ext}_{\rm CS} \ ,$$

of degree $|\psi_n^{\rm CS}| = 1 - n$ for $n \geq 1$. For $\psi_1^{\rm CS} : V^{\rm CS} \to V^{\rm ext}_{\rm CS}$, one uses the canonical embedding of the underlying vector space $V^{\rm CS}$ into $V^{\rm ext}_{\rm CS}$ to define a map of underlying cochain complexes

$$\begin{tikzcd}
 & V_0^{\rm CS} \arrow[r, "\ell_1^{\rm CS}"] \arrow[d, "\psi_1^{\rm CS}"'] & V_1^{\rm CS} \arrow[r, "\ell_1^{\rm CS}"] \arrow[d, "\psi_1^{\rm CS}"'] & V_2^{\rm CS} \arrow[r, "\ell_1^{\rm CS}"] \arrow[d, "\psi_1^{\rm CS}"'] & V_3^{\rm CS} \arrow[d, "\psi_1^{\rm CS}"'] & \\
V_{-1}^{\rm ext} \arrow[r, "\ell_1^{\rm ext}"] & V_0^{\rm ext} \arrow[r, "\ell_1^{\rm ext}"] & V_1^{\rm ext} \arrow[r, "\ell_1^{\rm ext}"] & V_2^{\rm ext} \arrow[r, "\ell_1^{\rm ext}"] & V_3^{\rm ext} \arrow[r, "\ell_1^{\rm ext}"] & V_4^{\rm ext}
\end{tikzcd}$$

given by

$$\psi_1^{\rm CS}(\lambda) = (0, \lambda) \ , \quad \psi_1^{\rm CS}(A) = A \ , \quad \psi_1^{\rm CS}(\mathcal{A}) = \mathcal{A} \qquad \text{and} \qquad \psi_1^{\rm CS}(\varLambda) = (0, \varLambda) \ .$$

For $\psi_2^{\rm CS} : \bigwedge^2 V^{\rm CS} \to V^{\rm ext}_{\rm CS}$, the only non-vanishing components are

$$\begin{aligned}
\psi_2^{\rm CS}(A, \varLambda) &:= -\big({\rm d}x^\mu \otimes {\rm Tr}_{\mathfrak{g}}(\iota_\mu {\rm d}\lambda \otimes \varLambda)\, ,\, 0\big) \ \in \ V_3^{\rm ext} \ , \\
\psi_2^{\rm CS}(\mathcal{A}_1, \mathcal{A}_2) &:= -\big({\rm d}x^\mu \otimes {\rm Tr}_{\mathfrak{g}}(\iota_\mu \mathcal{A}_1 \wedge \mathcal{A}_2)\, ,\, 0\big) \ \in \ V_3^{\rm ext} \ .
\end{aligned}$$

We set $\psi_n^{\rm CS} = 0$ for all $n \geq 3$.

It is easy to check that this defines an L_∞-morphism, as one sees by writing out the two sides of the relations (2.5). Furthermore, it is obviously a quasi-isomorphism, since $\psi_1^{\rm CS}$ descends to the identity on the corresponding cohomology groups $H^\bullet(V^{\rm CS}, \ell_1^{\rm CS}) = H^\bullet(V^{\rm ext}, \ell_1^{\rm ext})$, and is easily checked to be cyclic. In particular, this provides a simple example of an L_∞-morphism between differential graded Lie algebras which is not exactly a differential graded Lie algebra morphism, but only up to homotopy proportional to $\psi_2^{\rm CS}$: As a quasi-isomorphism of differential graded Lie algebras, it has an inverse only in the category of L_∞-algebras. This quasi-isomorphism provides a further way to see why infinitessimal diffeomorphisms are safely ignored in the pertubative quantisation of Chern-Simons.

4.4 BF theories in the L_∞-algebra formalism

The next primary example of a topological field theory with the same properties is a BF theory, and the exact analogue of the story spelled out thus far in this section can be easily adapted to BF theories in arbitrary dimension d and for any gauge group G. The BF action functional is given by

$$S_{\rm BF}(B,A) := \int_M \mathrm{Tr}_{\mathscr{W}}(B\wedge F) \tag{4.10}$$

where M is an oriented d-dimensional manifold, $F = \mathrm{d}^A A$ is the curvature of a connection one-form $A\in\Omega^1(M,\mathfrak{g})$ valued in a Lie algebra $\mathfrak{g}$, and B is a $(d-2)$-form valued in a fixed vector space $\mathscr{W}$ which is a $\mathfrak{g}$-module. The pairing $\mathrm{Tr}_{\mathscr{W}}:\mathscr{W}\otimes\mathfrak{g}\to\mathbb{R}$ is assumed to be invariant under the $\mathfrak{g}$-action:

$$\mathrm{Tr}_{\mathscr{W}}\big((X\cdot w)\otimes Y + w\otimes[X,Y]_{\mathfrak{g}}\big) = 0 \ ,$$

for $w\in\mathscr{W}$ and $X,Y\in\mathfrak{g}$. In the conventional definition of BF theory [46], one usually takes $\mathscr{W}=\mathfrak{g}$, so that both fields are valued in the same Lie algebra, with $\mathscr{W}$ regarded as a $\mathfrak{g}$-module under the canonical adjoint action of the Lie algebra on itself. However, this more general formulation will act as a nice stepping stone between different theories suitable for our purposes.

The field equations are readily seen to be

$$\begin{aligned}\mathcal{F}_B &:= F = \mathrm{d}A + \tfrac12\,[A,A]_{\mathfrak{g}} = 0 \ \in \ \Omega^2(M,\mathfrak{g}) \ ,\\ \mathcal{F}_A &:= \mathrm{d}^A B = \mathrm{d}B + A\wedge B = 0 \ \in \ \Omega^{d-1}(M,\mathscr{W}) \ ,\end{aligned}$$

where $A\wedge B$ computes the exterior product of the form components while pairing the components in $\mathfrak{g}$ and $\mathscr{W}$ via the $\mathfrak{g}$-action. Thus the Euler–Lagrange locus of BF theory are pairs of a flat G-connection on M and a covariantly constant $(d-2)$-form valued in a representation $\mathscr{W}$ of $\mathfrak{g}$.

The action functional (4.10) is invariant under standard (infinitesimal) gauge transformations $\rho\in\Omega^0(M,\mathfrak{g})$ acting as

$$\delta_\rho(B,A) = \big(-\rho\cdot B\,,\,\mathrm{d}\rho+[A,\rho]_{\mathfrak{g}}\big) \ .$$

Compared to Chern–Simons theory, however, for $d\geq 3$ there is the extra "shift" symmetry generated by $(d-3)$-forms $\tau\in\Omega^{d-3}(M,\mathscr{W})$ valued in $\mathscr{W}$, which act as

$$\delta_\tau(B,A) := (\mathrm{d}^A\tau,0) = (\mathrm{d}\tau + A\wedge\tau,0) \ . \tag{4.11}$$

This symmetry is on-shell reducible in dimensions $d\geq 4$. The corresponding Noether identities coincide with the usual "Bianchi identities"

$$\mathsf{d}_{(B,A)}(\mathcal{F}_B,\mathcal{F}_A) := \big((-1)^{d-3}\,\mathrm{d}^A\mathcal{F}_B\,,\,\mathrm{d}^A\mathcal{F}_A - \mathcal{F}_B\wedge B\big) = (0,0) \ \in \ \Omega^3(M,\mathfrak{g})\times\Omega^d(M,\mathscr{W}) \ ,$$

where the overall sign only serves to eliminate signs in the cyclic pairing and brackets below.

The cyclic L_∞-algebra of BF theory in d dimensions is given by the underlying graded vector space

$$V^{\rm BF} := V_0^{\rm BF}\oplus V_1^{\rm BF}\oplus V_2^{\rm BF}\oplus V_3^{\rm BF}$$

where

$$\begin{aligned}V_0^{\rm BF} &= \Omega^{d-3}(M,\mathscr{W})\times\Omega^0\big(M,\mathfrak{g}\big) \ ,\\ V_1^{\rm BF} &= \Omega^{d-2}(M,\mathscr{W})\times\Omega^1\big(M,\mathfrak{g}\big) \ ,\\ V_2^{\rm BF} &= \Omega^2\big(M,\mathfrak{g}\big)\times\Omega^{d-1}(M,\mathscr{W}) \ ,\\ V_3^{\rm BF} &= \Omega^3(M,\mathfrak{g})\times\Omega^d(M,\mathscr{W}) \ .\end{aligned} \tag{4.12}$$

We denote gauge parameters by $(\tau,\rho)\in V_0^{\rm BF}$, dynamical fields by $(B,A)\in V_1^{\rm BF}$, Euler–Lagrange derivatives by $(\mathcal{B},\mathcal{A})\in V_2^{\rm BF}$, and Noether identities by $(\mathcal{T},\mathcal{P})\in V_3^{\rm BF}$. The non-trivial brackets are then

$$
\begin{aligned}
\ell_1^{\rm BF}(\tau,\rho)&=({\rm d}\tau,{\rm d}\rho)\ \in\ V_1^{\rm BF}\ ,\\
\ell_1^{\rm BF}(B,A)&=({\rm d}A,{\rm d}B)\ \in\ V_2^{\rm BF}\ ,\\
\ell_1^{\rm BF}(\mathcal{B},\mathcal{A})&=({\rm d}\mathcal{B},{\rm d}\mathcal{A})\ \in\ V_3^{\rm BF}\ ,\\
\ell_2^{\rm BF}\big((\tau_1,\rho_1)\,,\,(\tau_2,\rho_2)\big)&=\big(-\rho_1\cdot\tau_2+\rho_2\cdot\tau_1\,,\,-[\rho_1,\rho_2]_{\mathfrak{g}}\big)\ \in\ V_0^{\rm BF}\ ,\\
\ell_2^{\rm BF}\big((\tau,\rho)\,,\,(B,A)\big)&=\big(-\rho\cdot B+A\wedge\tau\,,\,-[\rho,A]_{\mathfrak{g}}\big)\ \in\ V_1^{\rm BF}\ ,\\
\ell_2^{\rm BF}\big((\tau,\rho)\,,\,(\mathcal{B},\mathcal{A})\big)&=\big(-[\rho,\mathcal{B}]_{\mathfrak{g}}\,,\,-\rho\cdot\mathcal{A}+\mathcal{B}\wedge\tau\big)\ \in\ V_2^{\rm BF}\ ,\\
\ell_2^{\rm BF}\big((\tau,\rho)\,,\,(\mathcal{T},\mathcal{P})\big)&=\big(-[\rho,\mathcal{T}]_{\mathfrak{g}}\,,\,-\rho\cdot\mathcal{P}+(-1)^{d-3}\,\mathcal{T}\wedge\tau\big)\ \in\ V_3^{\rm BF}\ ,\\
\ell_2^{\rm BF}\big((B_1,A_1)\,,\,(B_2,A_2)\big)&=-\big([A_1,A_2]_{\mathfrak{g}}\,,\,A_1\wedge B_2+A_2\wedge B_1\big)\ \in\ V_2^{\rm BF}\ ,\\
\ell_2^{\rm BF}\big((B,A)\,,\,(\mathcal{B},\mathcal{A})\big)&=-\big([A,\mathcal{B}]_{\mathfrak{g}}\,,\,A\wedge\mathcal{A}-\mathcal{B}\wedge B\big)\ \in\ V_3^{\rm BF}\ ,
\end{aligned}
\tag{4.13}
$$

while all remaining brackets vanish. Thus the dynamics of BF theory in any dimension d is also organised by a differential graded Lie algebra. The cyclic pairing is given naturally as

$$
\begin{aligned}
\langle(B,A)\,,\,(\mathcal{B},\mathcal{A})\rangle^{\rm BF}&:=\int_M\,{\rm Tr}_{\mathscr{W}}(B\wedge\mathcal{B}+\mathcal{A}\wedge A)\ ,\\
\langle(\tau,\rho)\,,\,(\mathcal{T},\mathcal{P})\rangle^{\rm BF}&:=\int_M\,{\rm Tr}_{\mathscr{W}}(\tau\wedge\mathcal{T}+\rho\wedge\mathcal{P})\ .
\end{aligned}
$$

As with Chern–Simons theory, BF theory is invariant under infinitesimal diffeomorphisms, parameterized by vector fields in $\Gamma(TM)$, acting via the Lie derivative. However, these are again redundant: for any $\xi\in\Gamma(TM)$ we compute

$$
\begin{aligned}
\delta_\xi(B,A)&:=({\rm L}_\xi B,{\rm L}_\xi A)\\
&=({\rm d}\iota_\xi B+\iota_\xi{\rm d}B,{\rm d}\iota_\xi A+\iota_\xi{\rm d}A)\\
&=\big({\rm d}\tau_\xi+\iota_\xi{\rm d}B+\iota_\xi(A\wedge B)-\iota_\xi(A\wedge B)\,,\,{\rm d}\rho_\xi+\iota_\xi{\rm d}A+\iota_\xi[A,A]_{\mathfrak{g}}-\iota_\xi[A,A]_{\mathfrak{g}}\big)\\
&=\big({\rm d}\tau_\xi+\iota_\xi{\rm d}^AB+A\wedge\tau_\xi-\rho_\xi\cdot B\,,\,{\rm d}\rho_\xi+[A,\rho_\xi]_{\mathfrak{g}}+\iota_\xi{\rm d}^AA\big)\\
&=\big({\rm d}^A\tau_\xi-\rho_\xi\cdot B+\iota_\xi\mathcal{F}_A\,,\,{\rm d}^A\rho_\xi+\iota_\xi\mathcal{F}_B\big)\\
&=\delta_{(\tau_\xi,\rho_\xi)}(B,A)+(\iota_\xi\mathcal{F}_A\,,\,\iota_\xi\mathcal{F}_B)
\end{aligned}
\tag{4.14}
$$

where we defined $(\tau_\xi,\rho_\xi):=(\iota_\xi B,\iota_\xi A)\in V_0^{\rm BF}$. Thus again the diffeomorphism symmetry is redundant, as the further quotient of the moduli space of classical solutions $\mathscr{M}_{\rm BF}$ has no effect.

Of course, one may now augment the BF L_∞-algebra with this symmetry and its corresponding Noether identity, as we did for Chern–Simons theory. One can then follow the same procedure by adding the redundancy at $V_{-1}^{\rm ext}$ and its dual at $V_4^{\rm ext}$, and finally one ends up with a cyclic L_∞-algebra that is quasi-isomorphic to the one constructed here, which did not include the redundant symmetries to begin with. One also observes that the Einstein–Cartan–Palatini action functional (3.9) with $\Lambda=0$ is a special instance of the BF action functional (4.10) with $\mathfrak{g}=\mathfrak{so}(p,q)$, $\mathscr{W}=\bigwedge^{d-2}\mathbb{R}^{p,q}$ and the dynamical fields $(B,A)=(e^{d-2},\omega)$; however, this restriction of the field B to diagonal decomposable forms in $\Omega^{d-2}(M,\bigwedge^{d-2}\mathbb{R}^{p,q})$ breaks the shift symmetry of BF theory in

$d \geq 4$ dimensions. We will see how to interpret this observation in the L_∞-algebra framework later on.

Here we have worked in a local formulation of the field theory. There are different perspectives on what the global structure of the fields (B, A) should be. The usual approach is to view A as a connection on a principal G-bundle over M and B as a $(d-2)$-form valued in the vector bundle associated to the representation $\mathscr{W}$. In the case of non-trivial bundles, the diffeomorphism invariance has to be treated in a somewhat different way, and this will be explored in the special case of ECP gravity in Sections 5.3 and 5.4. There are also other interpretations, such as that of higher gauge theory which considers the fields as forms valued in a strict Lie 2-algebra [47], or equivalently in a 2-term L_∞-algebra. We shall not delve into this interpretation in the present paper, which however is interesting in view of the L_∞-algebra framework under consideration.

4.5 Higher shift symmetries

Another new feature of BF theories, compared to Chern–Simons theory, is that they possess additional redundant symmetries, in addition to diffeomorphisms: The shift symmetry (4.11) is on-shell reducible in dimensions $d \geq 4$. This means that, strictly speaking, we should also include in (4.12) the negatively-graded vector spaces $V^{\rm BF}_{-k} = \Omega^{d-3-k}(M, \mathscr{W})$ for $k = 1, \ldots, d-3$, which parameterize "higher gauge transformations", together with their duals $V^{\rm BF}_{k+3} = \Omega^{k+3}(M, \mathfrak{g})$ and the obvious brackets in (4.13). We spell out this out explicitly for BF theories in the simplest case $d = 4$.

Any element $\tau \in \Omega^1(M, \mathscr{W})$ generates the shift symmetry

$$\delta_\tau B = \mathrm{d}^A \tau \ \in \ \Omega^2(M, \mathscr{W}) \ .$$

Now take $\tau' := \tau + \mathrm{d}^A \epsilon \in \Omega^1(M, \mathscr{W})$ for any $\epsilon \in \Omega^0(M, \mathscr{W})$. Then this generates the shift symmetry

$$\begin{aligned} \delta_{\tau'} B &= \mathrm{d}^A \tau' \\ &= \mathrm{d}^A \tau + (\mathrm{d}^A)^2 \epsilon \\ &= \delta_\tau B + F \cdot \epsilon \\ &= \delta_\tau B + \mathcal{F}_B \cdot \epsilon \ , \end{aligned}$$

and so the two transformations differ by a term proportional to an Euler–Lagrange derivative, that is, τ and $\tau' =: \tau + \delta_{(\epsilon, A)} \tau$ generate the same symmetry on-shell. This leads to a further redundancy in the subspace of $V^{\rm BF}_0$ generating the distribution of shift symmetries on $V^{\rm BF}_1$; the redundancy lives in the subspace of covariantly exact one-forms valued in $\mathscr{W}$. We may parameterize this by $V^{\rm BF}_{-1} := \Omega^0(M, \mathscr{W})$; for $d = 4$ this is enough and no further gauge redundancy in the description exists, while in higher dimensions the form degrees change and similarly higher-to-higher gauge transformations are required, and so on.

Let us now describe the complete extended BF L_∞-algebra. We extend the cochain complex by introducing

$$V^{\rm BF}_{-1} := \Omega^0(M, \mathscr{W}) \qquad \text{and} \qquad V^{\rm BF}_4 := \Omega^4(M, \mathfrak{g}) \ ,$$

and denote the corresponding elements by $\epsilon \in V^{\rm BF}_{-1}$ and $\mathcal{E} \in V^{\rm BF}_4$. The brackets (4.13) are extended as

$$\ell^{\rm BF}_1(\epsilon) = (\mathrm{d}\epsilon, 0) \ \in \ V^{\rm BF}_0 \qquad \text{and} \qquad \ell^{\rm BF}_1(\mathcal{T}, \mathcal{P}) = \mathrm{d}\mathcal{T} \ \in \ V^{\rm BF}_4 \ ,$$

together with

$$
\begin{aligned}
\ell_2^{\rm BF}\big(\epsilon\,,\,(\tau,\rho)\big) &= \rho\cdot\epsilon \ \in\ V_{-1}^{\rm BF}\ ,\\
\ell_2^{\rm BF}\big(\epsilon\,,\,(B,A)\big) &= -(A\cdot\epsilon,0)\ \in\ V_0^{\rm BF}\ ,\\
\ell_2^{\rm BF}\big(\epsilon\,,\,(\mathcal{B},\mathcal{A})\big) &= -(\mathcal{B}\cdot\epsilon,0)\ \in\ V_1^{\rm BF}\ ,\\
\ell_2^{\rm BF}\big(\epsilon\,,\,(\mathcal{T},\mathcal{P})\big) &= -(0,\mathcal{T}\cdot\epsilon)\ \in\ V_2^{\rm BF}\ ,\\
\ell_2^{\rm BF}(\epsilon,\mathcal{E}) &= (0,\mathcal{E}\cdot\epsilon)\ \in\ V_3^{\rm BF}\ ,\\
\ell_2^{\rm BF}\big((\tau,\rho)\,,\,\mathcal{E}\big) &= -[\rho,\mathcal{E}]_{\mathfrak{g}}\ \in\ V_4^{\rm BF}\ ,\\
\ell_2^{\rm BF}\big((B,A)\,,\,(\mathcal{T},\mathcal{P})\big) &= [A,\mathcal{T}]_{\mathfrak{g}}\ \in\ V_4^{\rm BF}\ ,\\
\ell_2^{\rm BF}\big((\mathcal{B}_1,\mathcal{A}_1)\,,\,(\mathcal{B}_2,\mathcal{A}_2)\big) &= [\mathcal{B}_1,\mathcal{B}_2]_{\mathfrak{g}}\ \in\ V_4^{\rm BF}\ .
\end{aligned}
$$

The pairing extends naturally as

$$
\langle\epsilon,\mathcal{E}\rangle^{\rm BF} = \int_M \mathrm{Tr}_{\mathscr{W}}(\epsilon\wedge\mathcal{E})\ .
$$

All checks of the homotopy and cyclicity relations follow as before without any genuine novelty.

Observe in contrast with the redundancy of diffeomorphisms, one cannot simply pass to the subcomplex by “deleting” $V_{-1}^{\rm BF}$ and $V_4^{\rm BF}$ via a quasi-isomorphism since $H^{-1}(V^{\rm BF},\ell_1^{\rm BF}) = \mathbb{R}$ does not vanish. This is because there is no clear splitting of the gauge parameters $V_0^{\rm BF}$ into reducible and irreducible components, as it happens with diffeomorphisms. It is in this sense that the extented BF complex is crucial in the pertubative BV quantisation to fully resolve degeneracies, while diffeomorphisms may be safely ignored.

4.6 Three-dimensional BF and Chern–Simons theories

BF theories in three dimensions are particularly interesting in the present context. For $d=3$, the formulation of Section 4.4 is equivalent to the formulation of Section 4.1 for a specific Chern–Simons gauge theory: one takes the Lie algebra of the Chern–Simons theory to be the semi-direct product

$$
\hat{\mathfrak{g}} := \mathscr{W}\rtimes\mathfrak{g}\ ,
$$

where we view the vector space $\mathscr{W}$ as an abelian Lie algebra and use the action of $\mathfrak{g}$ on $\mathscr{W}$ to define the Lie bracket on $\hat{\mathfrak{g}}$. The invariant non-degenerate pairing on $\mathscr{W}\otimes\mathfrak{g}$ extends to $\hat{\mathfrak{g}}\otimes\hat{\mathfrak{g}}$, acting trivially on the $\mathfrak{g}\otimes\mathfrak{g}$ and $\mathscr{W}\otimes\mathscr{W}$ parts, and by symmetry to the rest. Then any gauge field $\hat{A}\in\Omega^1(M,\hat{\mathfrak{g}})$ has a unique decomposition

$$
\hat{A} = (B,A)
$$

with $B \in \Omega^1(M, \mathscr{W})$ and $A \in \Omega^1(M, \mathfrak{g})$. With these choices, some simple algebra shows that the Chern–Simons action functional coincides with the BF action functional in three dimensions:

$$
\begin{aligned}
S_{\rm CS}(\hat{A}) &= \frac{1}{2} \int_M \operatorname{Tr}_{\hat{\mathfrak{g}}}\Big(\hat{A} \wedge \mathrm{d}\hat{A} + \frac{1}{3}\,\hat{A} \wedge [\hat{A}, \hat{A}]_{\hat{\mathfrak{g}}}\Big) \\
&= \frac{1}{2} \int_M \operatorname{Tr}_{\hat{\mathfrak{g}}}\Big((B, A) \wedge (\mathrm{d}B, \mathrm{d}A) + \frac{1}{3}\,(B, A) \wedge \big(A \wedge B + A \wedge B, [A, A]_{\mathfrak{g}}\big)\Big) \\
&= \frac{1}{2} \int_M \operatorname{Tr}_{\mathscr{W}}\Big(2\,B \wedge \mathrm{d}A + \frac{2}{3}\,A \wedge (A \wedge B) + \frac{1}{3}\,B \wedge [A, A]_{\mathfrak{g}}\Big) \\
&= \int_M \operatorname{Tr}_{\mathscr{W}}\Big(B \wedge \mathrm{d}A + \frac{1}{2}\,B \wedge [A, A]_{\mathfrak{g}}\Big) \\
&= \int_M \operatorname{Tr}_{\mathscr{W}}(B \wedge F) \\
&= S_{\rm BF}(B, A) \ .
\end{aligned}
$$

The gauge symmetries map into each other as expected: the gauge transformation

$$
\delta_{(\tau,\rho)}(B, A) = \big(\delta_\tau B + \delta_\rho B\,,\, \delta_\tau A + \delta_\rho A\big) := \big(\mathrm{d}\tau + A \cdot \tau - \rho \cdot B\,,\, \mathrm{d}\rho + [A, \rho]_{\mathfrak{g}}\big)
$$

with $\hat{\lambda} = (\tau, \rho) \in \Omega^0(M, \hat{\mathfrak{g}})$ maps to

$$
\begin{aligned}
\delta_{\hat{\lambda}} \hat{A} &:= \mathrm{d}\hat{\lambda} + [\hat{A}, \hat{\lambda}]_{\hat{\mathfrak{g}}} \\
&= (\mathrm{d}\tau, \mathrm{d}\rho) + \big[(B, A)\,,\, (\tau, \rho)\big]_{\hat{\mathfrak{g}}} \\
&= \big(\mathrm{d}\tau + A \cdot \tau - \rho \cdot B\,,\, \mathrm{d}\rho + [A, \rho]_{\mathfrak{g}}\big) \ .
\end{aligned}
$$

The two field theories are essentially related by identity type redefinitions. The same holds of course at the level of the underlying L_∞-algebras, which are (strictly) isomorphic: the isomorphism is given by $\hat{\psi}_1 : V^{\rm BF} \to V^{\rm CS}$ as above, collecting the respective fields in each degree. In this sense, higher-dimensional BF theory is one way of generalizing Chern–Simons gauge theory to higher dimensions.

5 Einstein–Cartan–Palatini L_∞-algebras

We are now ready to move on to the main constructions of this paper: the cyclic L_∞-algebras underlying the ECP formalism from Section 3.

5.1 Brackets

We will first write down the general form of the L_∞-algebra structure for ECP gravity in an arbitrary dimensionality $d > 2$ discussed in Section 3 for the case when the spacetime M is parallelizable. Recalling the d-dimensional field equations (3.10) and Noether identities (3.11), the vector space V underlying the corresponding L_∞-algebra is

$$
V := V_0 \oplus V_1 \oplus V_2 \oplus V_3 \tag{5.1}
$$

where

$$V_0 = \Gamma(TM) \times \Omega^0\big(M, \mathfrak{so}(p,q)\big) \ ,$$

$$V_1 = \Omega^1(M, \mathbb{R}^{p,q}) \times \Omega^1\big(M, \mathfrak{so}(p,q)\big) \ ,$$

$$V_2 = \Omega^{d-1}\big(M, \textstyle\bigwedge^{d-1}(\mathbb{R}^{p,q})\big) \times \Omega^{d-1}\big(M, \textstyle\bigwedge^{d-2}(\mathbb{R}^{p,q})\big) \ ,$$

$$V_3 = \Omega^1\big(M, \Omega^d(M)\big) \times \Omega^d\big(M, \textstyle\bigwedge^{d-2}(\mathbb{R}^{p,q})\big) \ .$$

In the following we denote gauge parameters by $(\xi,\rho) \in V_0$, dynamical fields by $(e,\omega) \in V_1$, Euler–Lagrange derivatives by $(E,\varOmega) \in V_2$, and Noether identities by $(\mathcal{X},\mathcal{P}) \in V_3$.

The brackets on V may then be given as follows. The non-vanishing 1-brackets are defined by

$$\ell_1(\xi,\rho) = (0, \mathrm{d}\rho) \ \in \ V_1 \qquad \text{and} \qquad \ell_1(E,\varOmega) = (0, (-1)^{d-1}\, \mathrm{d}\varOmega) \ \in \ V_3 \ ,$$

while the non-vanishing 2-brackets are defined by

$$\begin{aligned}
\ell_2\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\big) &= \big([\xi_1,\xi_2]\,,\,-[\rho_1,\rho_2] + \xi_1(\rho_2) - \xi_2(\rho_1)\big) \ \in \ V_0 \ , \\
\ell_2\big((\xi,\rho)\,,\,(e,\omega)\big) &= \big(-\rho\cdot e + \mathrm{L}_\xi e\,,\,-[\rho,\omega] + \mathrm{L}_\xi\omega\big) \ \in \ V_1 \ , \\
\ell_2\big((\xi,\rho)\,,\,(E,\varOmega)\big) &= \big(-\rho\cdot E + \mathrm{L}_\xi E\,,\,-\rho\cdot\varOmega + \mathrm{L}_\xi\varOmega\big) \ \in \ V_2 \ , \\
\ell_2\big((\xi,\rho)\,,\,(\mathcal{X},\mathcal{P})\big) &= \big(\mathrm{d}x^\mu \otimes \mathrm{Tr}(\iota_\mu \mathrm{d}\rho \curlywedge \mathcal{P}) + \mathrm{L}_\xi\mathcal{X}\,,\,-\rho\cdot\mathcal{P} + \mathrm{L}_\xi\mathcal{P}\big) \ \in \ V_3 \ , \qquad (5.2) \\
\ell_2\big((e,\omega)\,,\,(E,\varOmega)\big) &= \Big(\mathrm{d}x^\mu \otimes \mathrm{Tr}\big(\iota_\mu\mathrm{d}e \curlywedge E + (-1)^{d-1}\,\iota_\mu\mathrm{d}\omega \curlywedge \varOmega - \iota_\mu e \curlywedge \mathrm{d}E - (-1)^{d-1}\,\iota_\mu\omega \curlywedge \mathrm{d}\varOmega\big)\,, \\
&\qquad\qquad \frac{d-1}{2}\,E \wedge e - (-1)^{d-1}\,\omega \wedge \varOmega\Big) \ \in \ V_3 \ .
\end{aligned}$$

The first bracket of (5.2) is simply the Lie bracket of the gauge algebra (3.6). The multivector action of a gauge transformation $\rho \in \Omega^0(M, \mathfrak{so}(p,q))$ on the Euler–Lagrange derivatives $(E,\varOmega)$ and on the Noether identity $\mathcal{P}$ is via the trivial coproduct $\Delta_0(\rho) = \rho\otimes 1 + 1\otimes\rho$ on the $\curlywedge$-products of fields, in the fundamental representation for e and in the adjoint representation for ω. For example, on $e_1 \curlywedge \mathrm{d}e_2 \in \Omega^3(M, \bigwedge^2(\mathbb{R}^{p,q}))$ the action is

$$\rho\cdot(e_1 \curlywedge \mathrm{d}e_2) = (\rho\cdot e_1) \curlywedge \mathrm{d}e_2 + e_1 \curlywedge (\rho\cdot\mathrm{d}e_2)$$

while on $e \curlywedge \mathrm{d}\omega \in \Omega^3(M, \bigwedge^3(\mathbb{R}^{p,q}))$ the action is

$$\rho\cdot(e \curlywedge \mathrm{d}\omega) = (\rho\cdot e) \curlywedge \mathrm{d}\omega + e \curlywedge [\rho, \mathrm{d}\omega] \ .$$

Similarly, the Lie derivative L_ξ acts on $\mathcal{X} \in \Omega^1(M)\otimes\Omega^d(M)$ via the Leibniz rule, that is, the trivial coproduct $\Delta_0(\mathrm{L}_\xi) = \mathrm{L}_\xi\otimes 1 + 1\otimes\mathrm{L}_\xi$.

Next, consider the brackets involving only dynamical fields. The $d{-}2$-bracket is defined by

$$\begin{aligned}
&\ell_{d-2}\big((e_1,\omega_1)\,,\,\dots\,,\,(e_{d-2},\omega_{d-2})\big) \\
&\qquad = (-1)^{\frac12\,(d-2)\,(d-3)} \sum_{\sigma\in S_{d-2}} \big(e_{\sigma(1)} \curlywedge e_{\sigma(2)} \curlywedge \cdots \curlywedge e_{\sigma(d-3)} \curlywedge \mathrm{d}\omega_{\sigma(d-2)}\,, \\
&\qquad\qquad\qquad\qquad e_{\sigma(1)} \curlywedge e_{\sigma(2)} \curlywedge \cdots \curlywedge \mathrm{d}e_{\sigma(d-2)}\big) \\
&\qquad = (-1)^{\frac12\,(d-2)\,(d-3)}\,(d-3)! \sum_{i=1}^{d-2} \big(e_1 \curlywedge \cdots \widehat{e_i} \cdots \curlywedge e_{d-2} \curlywedge \mathrm{d}\omega_i\,, \\
&\qquad\qquad\qquad\qquad e_1 \curlywedge \cdots \widehat{e_i} \cdots \curlywedge e_{d-2} \curlywedge \mathrm{d}e_i\big) \ \in \ V_2 \ .
\end{aligned}$$

The $d-1$-bracket is defined by

$$
\begin{aligned}
&\ell_{d-1}\big((e_1,\omega_1)\,,\,\dots\,,\,(e_{d-1},\omega_{d-1})\big)\\
&\quad= (-1)^{\frac12\,(d-1)\,(d-2)}\sum_{\sigma\in S_{d-1}}\Big(e_{\sigma(1)}\curlywedge\cdots e_{\sigma(d-3)}\curlywedge\tfrac12\,[\omega_{\sigma(d-2)},\omega_{\sigma(d-1)}]+\Lambda\,e_{\sigma(1)}\curlywedge\cdots\curlywedge e_{\sigma(d-1)}\,,\\
&\qquad\qquad\qquad\qquad\qquad e_{\sigma(1)}\curlywedge\cdots\curlywedge e_{\sigma(d-3)}\curlywedge\big(\omega_{\sigma(d-2)}\wedge e_{\sigma(d-1)}\big)\Big)\\
&\quad= (-1)^{\frac12\,(d-1)\,(d-2)}\,(d-3)!\sum_{\substack{i,j=1\\ i\neq j}}^{d-1}\Big(e_1\curlywedge\cdots\widehat{e_i\curlywedge e_j}\cdots\curlywedge e_{d-1}\curlywedge\tfrac12\,[\omega_i,\omega_j]\,,\\
&\qquad\qquad\qquad\qquad\qquad e_1\curlywedge\cdots\widehat{e_i\curlywedge e_j}\cdots\curlywedge e_{d-1}\curlywedge(\omega_i\wedge e_j)\Big)\\
&\qquad+(-1)^{\frac12\,(d-1)\,(d-2)}\,(d-1)!\,\Lambda\,(e_1\curlywedge\cdots\curlywedge e_{d-1},0)\ \in\ V_2\ .
\end{aligned}
\tag{5.3}
$$

The first expressions for the brackets ℓ_{d-2} and ℓ_{d-1} show that they are manifestly symmetric on V_1 as required. The second simplified expressions are obtained by noting that $e_i\curlywedge e_j=e_j\curlywedge e_i$. For $d>3$ the 1-bracket on fields is given by

$$
\ell_1(e,\omega)=(0,0)\ \in\ V_2\ ,
$$

while for $d>4$ the 2-bracket on fields is

$$
\ell_2\big((e_1,\omega_1)\,,\,(e_2,\omega_2)\big)=(0,0)\ \in\ V_2\ .
$$

The remaining brackets are all identically zero for all $d\geq 3$.

By construction, these brackets encode the gauge transformations, field equations and Noether identities of pure Einstein–Cartan–Palatini gravity in dimensions $d\geq 3$ as given in (2.11)–(2.15). For example, the gauge transformations are encoded by

$$
\begin{aligned}
\delta_{(\xi,\rho)}(e,\omega)&=\big(\delta_{(\xi,\rho)}e\,,\,\delta_{(\xi,\rho)}\omega\big)\\
&=\big(-\rho\cdot e+\mathrm{L}_\xi e\,,\,\mathrm{d}\rho-[\rho,\omega]+\mathrm{L}_\xi\omega\big)\\
&=\ell_1(\xi,\rho)+\ell_2\big((\xi,\rho)\,,\,(e,\omega)\big)\ \in\ V_1\ ,
\end{aligned}
$$

with the closure relation

$$
\big[\delta_{(\xi_1,\rho_1)}\,,\,\delta_{(\xi_2,\rho_2)}\big](e,\omega)=\delta_{-\ell_2((\xi_1,\rho_1),(\xi_2,\rho_2))}(e,\omega)
\tag{5.4}
$$

reflecting the module structure of the space of fields (3.5) for the Lie algebra of gauge transformations (3.6). Similarly, one also easily verifies

$$
\delta_{(\xi,\rho)}(\mathcal{F}_e,\mathcal{F}_\omega)=\ell_2\big((\xi,\rho)\,,\,(\mathcal{F}_e,\mathcal{F}_\omega)\big)\ \in\ V_2\ ,
$$

and the Noether identities are encoded through

$$
\mathsf{d}_{(e,\omega)}(\mathcal{F}_e,\mathcal{F}_\omega)=\ell_1(\mathcal{F}_e,\mathcal{F}_\omega)+\ell_2\big((\mathcal{F}_e,\mathcal{F}_\omega)\,,\,(e,\omega)\big)\ \in\ V_3\ .
$$

The quickest and most economical way to obtain these brackets is by bootstrapping [15]. One writes out the gauge transformations, Euler–Lagrange derivatives and Noether identities, and separates the orders of fields within the pairing as in Section 2.3. Then we demand that they are equal to a specific expansion in terms of linear, bilinear, trilinear, etc. brackets as in (2.11), (2.12) and (2.15), and one reads off the brackets by direct comparison together with the demand of cyclicity

(2.24). The extra non-zero brackets may be obtained by demanding that the homotopy relations hold. For example, the extra 2-bracket $\ell_2((\xi_1,\rho_1),(\xi_2,\rho_2))$ carries information about the Lie algebra structure of the gauge transformations, and of the action of $\Gamma(TM)$ on $\Omega^0(M,\mathfrak{so}(p,q))$. Indeed, this is the way we recovered the L_∞-algebra. The "disadvantage" of the approach is that one needs to check the homotopy relations explicitly. Another way to get the brackets is by developing and then dualizing the BV–BRST complex of the field theory [17], which has the advantage of automatically guaranteeing the homotopy relations, but at the cost of lengthy and cumbersome dualization calculations. The proof of the homotopy relations is tedious, but largely independent of the spacetime dimension d. We will explicitly prove the homotopy relations in Appendix A.1 in the simplest case $d=3$, the proof for higher dimensions being similar but requiring special care of the extra coframe field factors which are manifested in the form of higher brackets. We review and dualise the BV–BRST formalism of ECP developed by [24] for $d=4$ in Section 6, confirming the above L_∞-algebra structure.

5.2 Cyclic pairing

Given the brackets of Section 5.1, we wish to write the Einstein–Cartan–Palatini action functional (3.9) as in (2.24). For this, we need a suitable non-degenerate bilinear pairing $\langle -,-\rangle : V_1\otimes V_2\to\mathbb{R}$, which we shall show is given by

$$\langle (e,\omega)\,,\,(E,\varOmega)\rangle := \int_M \mathrm{Tr}\big(e\curlywedge E + (-1)^{d-1}\,\omega\curlywedge\varOmega\big) = \int_M \mathrm{Tr}\big(e\curlywedge E + \varOmega\curlywedge\omega\big)\ . \tag{5.5}$$

This can be extended to make (5.1) into a cyclic L_∞-algebra by introducing an additional pairing $\langle -,-\rangle : V_0\otimes V_3\to\mathbb{R}$ given by

$$\langle (\xi,\rho)\,,\,(\mathcal{X},\mathcal{P})\rangle := \int_M \iota_\xi\mathcal{X} + \int_M \mathrm{Tr}\big(\rho\curlywedge\mathcal{P}\big)\ . \tag{5.6}$$

The most general possible bilinear pairing could in principle include two arbitrary constants in front of each integrand. However, cyclicity demands they are set to unity, and we will now show that the pairings (5.5) and (5.6) indeed have the right cyclicity properties. The only non-trivial checks required in (2.22) are for the brackets ℓ_{d-2} and ℓ_{d-1}. The explicit demonstration of the cyclicity (2.23) for the pairing (5.5) is elementary as it involves only the bracket ℓ_2 from (5.2).

Let us first establish cyclicity with respect to the bracket ℓ_{d-2}.

Lemma 5.7. *If $(e_i,\omega_i)\in V_1$ for $i=0,1,\dots,d-2$, then*

$$\big\langle (e_0,\omega_0)\,,\,\ell_{d-2}\big((e_1,\omega_1),\dots,(e_{d-2},\omega_{d-2})\big)\big\rangle = \big\langle (e_1,\omega_1)\,,\,\ell_{d-2}\big((e_0,\omega_0),(e_2,\omega_2),\dots,(e_{d-2},\omega_{d-2})\big)\big\rangle\ .$$

Proof. We shall ignore the overall constants since they are the same on both sides of this equality. Thus writing out the left-hand side we obtain

$$\begin{aligned}
&\int_M \mathrm{Tr}\Big(\sum_{i=1}^{d-2}\big(e_0\curlywedge e_1\curlywedge\cdots\widehat{e_i}\cdots\curlywedge e_{d-2}\curlywedge \mathrm{d}\omega_i + (-1)^{d-1}\,\omega_0\curlywedge e_1\curlywedge\cdots\widehat{e_i}\cdots\curlywedge e_{d-2}\curlywedge\mathrm{d}e_i\big)\Big)\\
&\quad= \int_M \mathrm{Tr}\Big(e_0\curlywedge e_2\curlywedge\cdots\curlywedge e_{d-2}\curlywedge\mathrm{d}\omega_1 + \sum_{i=2}^{d-2}\big(e_0\curlywedge e_1\curlywedge\cdots\widehat{e_i}\cdots\curlywedge e_{d-2}\curlywedge\mathrm{d}\omega_i\big)\\
&\qquad\qquad + (-1)^{d-1}\,\omega_0\curlywedge e_2\curlywedge\cdots\curlywedge e_{d-2}\curlywedge\mathrm{d}e_1\\
&\qquad\qquad\qquad + (-1)^{d-1}\sum_{i=2}^{d-2}\big(\omega_0\curlywedge e_1\cdots\widehat{e_i}\cdots\curlywedge e_{d-2}\curlywedge\mathrm{d}e_i\big)\Big)\ .
\end{aligned}\tag{5.8}$$

Integrating by parts on the first and third terms, and dropping exact forms since we only consider coframes with compact support, we get

$$
\begin{aligned}
\int_M \mathrm{Tr}\Big(&(-1)^{d-3}\, \omega_1 \curlywedge e_2 \curlywedge \cdots \curlywedge e_{d-2} \curlywedge \mathrm{d}e_0 - \omega_1 \curlywedge e_0 \curlywedge \mathrm{d}(e_2 \curlywedge \cdots \curlywedge e_{d-2}) \\
&+ \sum_{i=2}^{d-2} \big(e_0 \curlywedge e_1 \curlywedge \cdots \widehat{e_i} \cdots \curlywedge e_{d-2} \curlywedge \mathrm{d}\omega_i\big) + \omega_0 \curlywedge e_1 \curlywedge \mathrm{d}(e_2 \curlywedge \cdots \curlywedge e_{d-2}) \\
&+ e_1 \curlywedge e_2 \curlywedge \cdots \curlywedge e_{d-2} \curlywedge \mathrm{d}\omega_0 + (-1)^{d-1} \sum_{i=2}^{d-2} \big(\omega_0 \curlywedge e_1 \curlywedge \cdots \widehat{e_i} \cdots \curlywedge e_{d-2} \curlywedge \mathrm{d}e_i\big)\Big) \ .
\end{aligned}
$$

Now using

$$
\omega_0 \curlywedge e_1 \curlywedge \mathrm{d}(e_2 \curlywedge \cdots \curlywedge e_{d-2}) = -(-1)^{d-3} \sum_{i=2}^{d-2} \big(\omega_0 \curlywedge e_1 \curlywedge \cdots \widehat{e_i} \cdots \curlywedge e_{d-2} \curlywedge \mathrm{d}e_i\big)
$$

we see that the fourth and sixth terms cancel. Substituting similarly for the second term, we get

$$
\begin{aligned}
\int_M \mathrm{Tr}\Big(&(-1)^{d-1}\, \omega_1 \curlywedge e_2 \curlywedge \cdots \curlywedge e_{d-2} \curlywedge \mathrm{d}e_0 + (-1)^{d-1} \sum_{i=2}^{d-2} \big(\omega_1 \curlywedge e_0 \curlywedge \cdots \widehat{e_i} \cdots \curlywedge e_{d-2} \curlywedge \mathrm{d}e_i\big) \\
&+ \sum_{i=2}^{d-2} \big(e_1 \curlywedge e_0 \curlywedge \cdots \widehat{e_i} \cdots \curlywedge e_{d-2} \curlywedge \mathrm{d}\omega_i\big) + e_1 \curlywedge e_2 \curlywedge \cdots \curlywedge e_{d-2} \curlywedge \mathrm{d}\omega_0 \Big) \ .
\end{aligned}
$$

This is just the equality (5.8) with the indices 1 and 0 interchanged, showing that the pairing is indeed cyclic under ℓ_{d-2} as claimed. $\square$

Next we establish cyclicity with respect to the bracket ℓ_{d-1}.

Lemma 5.9. *If $(e_i, \omega_i) \in V_1$ for $i = 0, 1, \dots, d-1$, then*

$$
\big\langle (e_0,\omega_0)\,,\, \ell_{d-1}\big((e_1,\omega_1),\dots,(e_{d-1},\omega_{d-1})\big)\big\rangle = \big\langle (e_1,\omega_1)\,,\, \ell_{d-1}\big((e_0,\omega_0),(e_2,\omega_2),\dots,(e_{d-1},\omega_{d-1})\big)\big\rangle \ .
$$

Proof. Ignoring again overall prefactors, we compute

$$
\begin{aligned}
&\big\langle (e_0,\omega_0)\,,\, \ell_{d-1}\big((e_1,\omega_1),\dots,(e_{d-1},\omega_{d-1})\big)\big\rangle \\
&\qquad = \int_M \mathrm{Tr}\Big(\sum_{\substack{i,j=1 \\ i\neq j}}^{d-1} \big(e_0 \curlywedge e_1 \curlywedge \cdots \widehat{e_i \curlywedge e_j} \cdots \curlywedge e_{d-1} \curlywedge \tfrac{1}{2}\,[\omega_i,\omega_j]\big) \\
&\qquad\qquad + (-1)^{d-1} \sum_{\substack{i,j=1 \\ i\neq j}}^{d-1} \big(\omega_0 \curlywedge e_1 \curlywedge \cdots \widehat{e_i \curlywedge e_j} \cdots \curlywedge e_{d-1} \curlywedge (\omega_i \wedge e_j)\big)\Big) \ .
\end{aligned}
$$

Here we dropped the cosmological constant term as it is easily seen to contribute cyclically to this pairing, since $e_0 \curlywedge e_1 = e_1 \curlywedge e_0$. We wish to write this in a manifestly symmetric form under the exchange of the indices 1 and 0. Since $[\omega_i,\omega_j]=[\omega_j,\omega_i]$, the first term may be rewritten as

$$
\begin{aligned}
&\sum_{\substack{i,j=1\\ i\neq j}}^{d-1} \big(e_0 \curlywedge e_1 \curlywedge \cdots \widehat{e_i \curlywedge e_j} \cdots \curlywedge e_{d-1} \curlywedge \tfrac{1}{2}\,[\omega_i,\omega_j]\big) \\
&\qquad = 2\sum_{\substack{i,j=1\\ i<j}}^{d-1} \big(e_0 \curlywedge e_1 \curlywedge \cdots \widehat{e_i \curlywedge e_j} \cdots \curlywedge e_{d-1} \curlywedge \tfrac{1}{2}\,[\omega_i,\omega_j]\big) \\
&\qquad = \sum_{\substack{i,j=2\\ i<j}}^{d-1} \big(e_0 \curlywedge e_1 \curlywedge \cdots \widehat{e_i \curlywedge e_j} \cdots \curlywedge e_{d-1} \curlywedge [\omega_i,\omega_j]\big) \\
&\qquad\quad + \sum_{j=2}^{d-1} \big(e_0 \curlywedge \widehat{e_1} \cdots \widehat{e_j} \cdots \curlywedge e_{d-1} \curlywedge [\omega_1,\omega_j]\big)\ .
\end{aligned}
$$

For the second term, we use $\omega \curlywedge e = -e \curlywedge \omega$ to get

$$
\begin{aligned}
&(-1)^{d-1}\,(-1)^{d-3} \sum_{\substack{i,j=1\\ i\neq j}}^{d-1} \big(e_1 \curlywedge \cdots \widehat{e_i \curlywedge e_j} \cdots \curlywedge e_{d-1} \curlywedge (\omega_i \wedge e_j) \curlywedge \omega_0\big) \\
&\qquad = \sum_{\substack{i,j=1\\ i\neq j}}^{d-1} \big(e_1 \curlywedge \cdots \widehat{e_i \curlywedge e_j} \cdots \curlywedge e_{d-1} \curlywedge (\omega_i \wedge e_j) \curlywedge \omega_0\big)\ .
\end{aligned}
$$

Next we use the identity (3.8) to write this as

$$
\begin{aligned}
\sum_{i=1}^{d-1} \big(e_1 \curlywedge \cdots \widehat{e_i} \cdots \curlywedge e_{d-1} \curlywedge [\omega_i,\omega_0]\big) &= \sum_{i=2}^{d-1} \big(e_1 \curlywedge \cdots \widehat{e_i} \cdots \curlywedge e_{d-1} \curlywedge [\omega_i,\omega_0]\big) \\
&\quad + e_2 \curlywedge \cdots \curlywedge e_{d-1} \curlywedge [\omega_1,\omega_0]\ .
\end{aligned}
$$

Finally collecting everything together, we get

$$
\begin{aligned}
&\big\langle (e_0,\omega_0)\,,\ell_{d-1}\big((e_1,\omega_1),\dots,(e_{d-1},\omega_{d-1})\big)\big\rangle \\
&\quad = \int_M \mathrm{Tr}\bigg(\sum_{\substack{i,j=2\\ i<j}}^{d-1} \big(e_0 \curlywedge e_1 \curlywedge \cdots \widehat{e_i \curlywedge e_j} \cdots \curlywedge e_{d-1} \curlywedge [\omega_i,\omega_j]\big) \\
&\qquad\qquad + \sum_{j=2}^{d-1} \big(e_0 \curlywedge \widehat{e_1} \cdots \widehat{e_j} \cdots \curlywedge e_{d-1} \curlywedge [\omega_1,\omega_j]\big) \\
&\qquad\qquad + \sum_{i=2}^{d-1} \big(e_1 \curlywedge \cdots \widehat{e_i} \cdots \curlywedge e_{d-1} \curlywedge [\omega_i,\omega_0]\big) + e_2 \curlywedge \cdots \curlywedge e_{d-1} \curlywedge [\omega_1,\omega_0]\bigg)\ .
\end{aligned}
$$

This expression is manifestly symmetric under exchange of the indices 1 and 0: The first term is invariant since $e_1 \curlywedge e_0 = e_0 \curlywedge e_1$, the last term is unchanged since $[\omega_1,\omega_0]=[\omega_0,\omega_1]$, and the remaining terms map into each other under the exchange. This completes the proof of cyclicity of the pairing (5.5). $\square$

Now we establish cyclicity with respect to the bracket ℓ_2. We only exhibit the non-trivial check, with the rest following similarly.

Lemma 5.10. *If* $(\xi_i,\rho_i)\in V_0$ *for* $i=0,1$ *and* $(\mathcal{X},\mathcal{P})\in V_3$*, then*

$$\big\langle(\xi_0,\rho_0)\,,\,\ell_2\big((\xi_1,\rho_1),(\mathcal{X},\mathcal{P})\big)\big\rangle=-\big\langle(\xi_1,\rho_1)\,,\,\ell_2\big((\xi_0,\rho_0),(\mathcal{X},\mathcal{P})\big)\big\rangle=\big\langle(\mathcal{X},\mathcal{P})\,,\,\ell_2\big((\xi_0,\rho_0),(\xi_1,\rho_1)\big)\big\rangle\ .$$

Proof. By linearity, it suffices to prove the result for decomposable $\mathcal{X}=\mathcal{X}_1\otimes\mathcal{X}_d\in\Omega^1(M,\Omega^d(M))$ with $\mathcal{X}_1$ a one-form and $\mathcal{X}_d$ a d-form on M. Then

$$\begin{aligned}\int_M\iota_{\xi_0}\mathrm{L}_{\xi_1}\mathcal{X}&=\int_M\big(\iota_{\xi_0}\mathrm{L}_{\xi_1}\mathcal{X}_1\,\mathcal{X}_d+\iota_{\xi_0}\mathcal{X}_1\,\mathrm{d}\iota_{\xi_1}\mathcal{X}_d\big)\\&=\int_M\big(\iota_{\xi_0}\mathrm{L}_{\xi_1}\mathcal{X}_1\,\mathcal{X}_d-\mathrm{d}\iota_{\xi_0}\mathcal{X}_1\wedge\iota_{\xi_1}\mathcal{X}_d\big)\\&=\int_M\big(\iota_{\xi_0}\mathrm{L}_{\xi_1}\mathcal{X}_1\,\mathcal{X}_d-\iota_{\xi_1}\mathrm{d}\iota_{\xi_0}\mathcal{X}_1\,\mathcal{X}_d\big)\\&=\int_M\big(\iota_{\xi_0}\mathrm{L}_{\xi_1}\mathcal{X}_1\,\mathcal{X}_d-\mathrm{L}_{\xi_1}\iota_{\xi_0}\mathcal{X}_1\,\mathcal{X}_d\big)\\&=\int_M\iota_{[\xi_0,\xi_1]}\mathcal{X}\ ,\end{aligned}$$

where we firstly used the trivial coproduct to distribute the Lie derivative, then applied Cartan's magic formula, integrated by parts and used the derivation property of the contraction. Lastly we used the Cartan identity

$$\iota_{[\xi_1,\xi_0]}=\mathrm{L}_{\xi_1}\circ\iota_{\xi_0}-\iota_{\xi_0}\circ\mathrm{L}_{\xi_1}\ . \tag{5.11}$$

The final equality says that the initial quantity on the left-hand side is antisymmetric under the exchange of the vector fields ξ_0 and ξ_1. Using this, the left-hand side of the cyclicity identity expands as

$$\begin{aligned}\big\langle(\xi_0,\rho_0)\,,\,\ell_2\big((\xi_1,\rho_1),(\mathcal{X},\mathcal{P})\big)\big\rangle&=\big\langle(\xi_0,\rho_0)\,,\,\big(\mathrm{d}x^\mu\otimes\mathrm{Tr}(\iota_\mu\mathrm{d}\rho_1\curlywedge\mathcal{P})+\mathrm{L}_{\xi_1}\mathcal{X},-\rho_1\cdot\mathcal{P}+\mathrm{L}_{\xi_1}\mathcal{P}\big)\big\rangle\\&=\int_M\iota_{[\xi_0,\xi_1]}\mathcal{X}+\int_M\mathrm{Tr}\big(\iota_{\xi_0}\mathrm{d}\rho_1\curlywedge\mathcal{P}-\rho_0\curlywedge\rho_1\cdot\mathcal{P}+\rho_0\curlywedge\mathrm{L}_{\xi_1}\mathcal{P}\big)\\&=\int_M\iota_{[\xi_0,\xi_1]}\mathcal{X}+\int_M\mathrm{Tr}\big(-\rho_1\curlywedge\mathrm{L}_{\xi_0}\mathcal{P}+[\rho_1,\rho_0]\curlywedge\mathcal{P}+\rho_0\curlywedge\mathrm{L}_{\xi_1}\mathcal{P}\big)\end{aligned}$$

where we used used the derivation propery of the contraction, integrated by parts and used the invariance of a top exterior vector in $\mathbb{R}^{p,q}$ under $\mathfrak{so}(p,q)$ rotations. This is manifestly antisymmetric under the exchange of indices 0 and 1, thus proving the first cyclicity identity. Further manipulating, noting that $\iota_{\xi_0}\mathrm{d}\rho_1=\mathrm{L}_{\xi_0}\rho_1$ since ρ_1 is a zero-form, this is also equal to

$$\int_M\iota_{[\xi_0,\xi_1]}\mathcal{X}+\int_M\mathrm{Tr}\big((\mathrm{L}_{\xi_0}\rho_1-\mathrm{L}_{\xi_1}\rho_0)\curlywedge\mathcal{P}-[\rho_0,\rho_1]\curlywedge\mathcal{P}\big)$$

which gives the final cyclicity identity. □

Finally, the proof of the cyclicity of $\big\langle(\xi,\rho)\,,\,\ell_2\big((e,\omega),(E,\Omega)\big)\big\rangle$ contains essentially no new ideas, apart from the fact that $\rho\cdot e\curlywedge E=-\frac{d-1}{2}\,\rho\curlywedge(E\wedge e)$, which follows from $0=(\rho\curlywedge E)\wedge e$ since the internal vector space exterior products combine to a $d{+}1$-form in d dimensions, and then distributing the contractions. To summarise, we have completely encoded the dynamical content of pure ECP gravity in $d>2$ dimensions in terms of an L_∞-algebra equipped with a suitable cyclic pairing. In Sections 7 and 8 we will consider two physically relevant examples explicitly to illustrate this structure.

5.3 Covariant L_∞-algebra

We will now discuss the covariance problems associated with our L_∞-algebra for the ECP theory, and how to deal with them. This is done using the well known covariant Lie derivative [48] [49] [50], whose geometric meaning we review and meet its avatar in the resulting brackets. The resulting L_∞-algebra is completely dual to the covariant version of the BV-BRST complex developed in [24] for $d = 4$.

Finite gauge transformations

Recall from Sections 3.1 and 3.2 that in order for these formulas to make sense, one firstly has to consider a parallelizable spacetime manifold M and fix the 'fake tangent bundle' $\mathscr{V} = M \times \mathbb{R}^{p,q}$.[22] Under this choice, the coframe field e may be viewed globally as a one-form on M valued in $\mathbb{R}^{p,q}$ and the connection ω as a one-form valued in $\mathfrak{so}(p,q)$. For our original motivation on parallelisable manifolds and noncommutative or nonassociative deformations this suffices [28], however one runs into issues if finite gauge transformations and the possible non-parallelisability of spacetime are taken into account. The coframe is globally encoded as a one-form $\tilde{e} \in \Omega^1(\mathscr{P}, \mathbb{R}^{p,q})$ and the connection is globally encoded as a one-form $\tilde{\omega} \in \Omega^1(\mathscr{P}, \mathfrak{so}(p,q))$, on the associated principal $\mathsf{SO}_+(p,q)$-bundle $\mathscr{P} \to M$. Given a local trivialisation of $\mathscr{P}$, or equivalently a local section $s : U \to \mathscr{P}$ for $U \subset M$, one defines the gauge field $\omega := s^*\tilde{\omega} \in \Omega^1(U, \mathfrak{so}(p,q))$. Given another local section $s' : U \to \mathscr{P}$ with $\omega' := s'^*\tilde{\omega}$, the two pullbacks are related by

$$\omega' = h^{-1}\, \omega\, h + h^{-1}\, \mathrm{d}h$$

where $h : U \to \mathsf{SO}_+(p,q)$ is the finite gauge transformation defined by $s' = s\, h$.

The "problem" arises with the diffeomorphism symmetry of the theory: given an infinitessimal diffeomorphism of the base manifold M, parameterized by a vector field $\xi \in \Gamma(TM)$, it is clear that

$$\omega' + \mathrm{L}_\xi \omega' \neq h^{-1}\, (\omega + \mathrm{L}_\xi \omega)\, h + h^{-1}\, \mathrm{d}h$$

for any $h : U \to \mathsf{SO}_+(p,q)$, and furthermore one cannot identify any section of $\mathscr{P}$ which pulls back $\tilde{\omega}$ to $\omega + \mathrm{L}_\xi \omega$. That is, the expressions $\omega + \mathrm{L}_\xi \omega$ and $\omega' + \mathrm{L}_\xi \omega'$ no longer define a connection on the same bundle $\mathscr{P}$. This is apparent if one uses finite diffeomorphisms $\phi : M \to M$ of the base, where the pullbacks of the gauge fields $\phi^*\omega$ and $\phi^*\omega'$ do define a connection, but on the pullback principal bundle $\phi^*\mathscr{P}$ over M instead. Thus, strictly speaking, our approach applies only when one completely disregards global structures and thus also finite gauge transformations, viewing the fields as globally defined defined one-forms on the base space $\omega \in \Omega^1(M, \mathfrak{so}(p,q))$ and $e \in \Omega^1(M, \mathbb{R}^{p,q})$ which transform only under infinitesimal gauge transformations as expected.

Covariantization of diffeomorphisms

We have seen that a Lie derivative of the gauge field on the base space no longer defines a connection on the same principal bundle, even if the bundle is trivial, and similarly for the coframe field when viewed as a section of $T^*M \otimes \mathscr{V}$. An equivalent way to spot the issue is from the fact that the action of the Lie derivative does not commute with the action of a finite gauge transformation $h : U \to \mathsf{SO}_+(p,q)$; for example, on the coframe field

$$\mathrm{L}_\xi (h^{-1}\, e) \neq h^{-1}\, \mathrm{L}_\xi e \ .$$

[22] Alternatively, they make sense in any fixed local trivialization of $\mathscr{V}$, but then one has to face the problem of patching together the locally defined L_∞-algebras in a suitable way to an 'L_∞-algebroid stack' on the spacetime M. In the present paper we instead follow a more concrete approach to this problem, which is detailed in the following.

Equivalently, for an infinitesimal pseudo-orthogonal rotation $\rho : M \to \mathfrak{so}(p,q)$,

$$[\delta_\xi, \delta_\rho] \neq 0 \ ,$$

as was already implied by the semi-direct product structure of the gauge algebra (3.6).

The resolution comes by identifying the correct way to act directly on the global fields $\tilde{e}$, $\tilde{\omega}$ living on $\mathscr{P}$, where they appear as genuine one-forms valued in fixed vector spaces. We briefly describe this and skip the straightforward differential geometrical calculations, since they are well known and not relevant for the rest of the section. Each connection identifies a horizontal distribution $\mathrm{Hor}(\mathscr{P}) \subset T\mathscr{P}$ via its kernel, splitting the tangent bundle $T\mathscr{P} = \mathrm{Vert}(\mathscr{P}) \oplus \mathrm{Hor}(\mathscr{P})$ such that $\mathrm{Hor}(\mathscr{P}) \cong TM$ via the differential of the bundle projection $\pi : \mathscr{P} \to M$ and $\mathrm{Vert}(\mathscr{P}) \equiv \ker(\mathrm{d}\pi)$. Using this identification, $\Gamma(TM) \cong \Gamma(\mathrm{Hor}(\mathscr{P}))$ as vector spaces, and so we may act with the unique lift $\tilde{\xi} \in \Gamma(\mathrm{Hor}(\mathscr{P}))$ of any $\xi \in \Gamma(TM)$, via the Lie derivative of the total space instead. Notice although the connection gives a lift $\Gamma(TM) \cong \Gamma(\mathrm{Hor}(\mathscr{P}))$, this is not a Lie algebra morphism, that is

$$[\tilde{\xi_1}, \tilde{\xi_2}] = \widetilde{[\xi_1, \xi_2]} + \iota_{\tilde{\xi}_2} \iota_{\tilde{\xi}_1} \tilde{R} \tag{5.12}$$

expressing that the non-integrability of the horizontal distribution is controlled by the curvature of the connection. Acting on the global fields $(\tilde{e}, \tilde{\omega})$ using the equivariance, horizontal and vertical properties of the fields along with Cartan calculus on $\mathscr{P}$,

$$(\mathrm{L}_{\tilde{\xi}} \tilde{e}, \mathrm{L}_{\tilde{\xi}} \tilde{\omega}) = \big((\mathrm{d}^{\tilde{\omega}} \circ \iota_{\tilde{\xi}} + \iota_{\tilde{\xi}} \circ \mathrm{d}^{\tilde{\omega}})\, \tilde{e} \, , \, \iota_{\tilde{\xi}} \tilde{R}\big)$$

where the right hand side is manifestly horizontal and equivariant. Hence $(\tilde{e}, \tilde{\omega}) + (\mathrm{L}_{\tilde{\xi}} \tilde{e}, \mathrm{L}_{\tilde{\xi}} \tilde{\omega})$ define a coframe and a connection as expected. Using a local section $s : U \to \mathscr{P}$ and equivariance, the expressions pull down to define the infinitessimal action we are after

$$\delta^{\mathrm{cov}}_\xi (e, \omega) := \big(\mathrm{L}^\omega_\xi e \, , \, \iota_\xi R\big) = \big(\mathrm{L}_\xi e + \iota_\xi \omega \cdot e \, , \, \iota_\xi \mathrm{d}\omega + [\iota_\xi \omega, \omega]\big) \ , \tag{5.13}$$

where the *covariant Lie derivative*

$$\mathrm{L}^\omega_\xi := \mathrm{d}^\omega \circ \iota_\xi + \iota_\xi \circ \mathrm{d}^\omega$$

is defined on the spacetime as an appropriate modification of the Cartan formula (3.4). Notice, due to (5.12) this does not form a Lie algebra action of $\Gamma(TM)$ on the space of fields; however, we will see the L_∞-algebra framework is sufficient to accomodate such situations. As a further check, one may confirm by working directly on the base spacetime manifold that the above infinitesimal action is indeed covariant, i.e. it commutes with local pseudo-orthogonal rotations:

$$[\delta^{\mathrm{cov}}_\xi, \delta_\rho](e, \omega) = (0, 0) \ .$$

In particular, they commute with finite pseudo-orthogonal rotations which are connected to the identity, that is, the fields $(e, \omega) + \delta^{\mathrm{cov}}_\xi (e, \omega)$ then do form a proper section of a vector bundle and a connection on its associated principal bundle, as expected from the total space formulation. Obviously, the above discussion applies for gauge fields with any internal group G and for matter fields valued in any G-representation. As such, covariant Lie derivatives (also known as 'covariant general coordinate transformations') have appeared in various contexts, to name a few: Studying symmetries and conserved quantities of gauge theories on fixed background spacetimes [48]; More specifically, these produce the correct symmetric and gauge invariant energy-momentum tensor in Minkowski spacetime, avoiding the "adhoc" Belinfante procedure. They also appear necessarily in the closure of local supersymmetry transformations in supergravity [49]. Furthermore, they have been recently used to study black hole thermodynamic laws in the case where non-trivial bundle topologies underlie the dynamical fields [50].

The action functional (3.9) is indeed invariant under the new covariant diffeomorphisms. For example, one can check

$$\delta_\xi^{\rm cov}(e^{d-2} \curlywedge R) = {\rm L}_\xi(e^{d-2} \curlywedge R) + \iota_\xi\omega \cdot (e^{d-2} \curlywedge R)$$

where the first term vanishes upon integration over M by the usual diffeomorphism invariance, and the second term vanishes by invariance under local pseudo-orthogonal transformations. Both at the field transformation and at the action functional level we see the two diffeomorphism actions differ by a local rotation, thus they are equivalent. We will see this equivalence translates to the corresponding cyclic L_∞-algebras being isomorphic. Expressing $\delta_\xi^{\rm cov} S_{\rm ECP} = 0$ through

$$\delta_\xi^{\rm cov} S_{\rm ECP} = \int_M {\rm Tr}\big(\mathcal{F}_e \curlywedge \delta_\xi^{\rm cov} e + \mathcal{F}_\omega \curlywedge \delta_\xi^{\rm cov}\omega\big)$$

and isolating ξ as previously, the Noether identity corresponding to the covariant diffeomorphism modifies only the first component of (3.12) to

$$\begin{aligned}\mathsf{d}^{\rm cov}_{(e,\omega)}(\mathcal{F}_e, \mathcal{F}_\omega) := \bigg(&{\rm d}x^\mu \otimes {\rm Tr}\Big(\iota_\mu e \curlywedge {\rm d}\mathcal{F}_e - \iota_\mu {\rm d}e \curlywedge \mathcal{F}_e - (-1)^{d-1}\,\iota_\mu {\rm d}\omega \curlywedge \mathcal{F}_\omega \\ &+ \iota_\mu\omega \curlywedge \Big[\frac{d-1}{2}\,\mathcal{F}_e \wedge e - (-1)^{d-1}\,\omega \wedge \mathcal{F}_\omega\Big]\Big)\,,\ -\frac{d-1}{2}\,\mathcal{F}_e \wedge e + (-1)^{d-1}\,{\rm d}^\omega \mathcal{F}_\omega\bigg)\ .\end{aligned} \tag{5.14}$$

Covariant brackets

We shall now spell out the brackets of the bootstrapped L_∞-algebra corresponding to the covariant gauge transformations (5.13). The L_∞-algebra we obtain turns out to be dual to the covariant BV differential obtained in [24] for the case of $d = 4$. The brackets involving solely dynamical fields and the Euler–Lagrange derivatives are the same as those of Section 5.1, as only the gauge transformations and Noether identities are affected in the covariant formulation, but not the dynamics. The underlying vector space is locally as before, however we now consider the possibly non-trivial bundle structures properly. For this, we parameterize $\mathsf{SO}_+(p,q)$-connections on the principal bundle $\mathscr{P} \to M$ by one-forms $\Omega^1\big(M, \mathscr{P} \times_{\rm ad} \mathfrak{so}(p,q)\big)$ on the base M valued in the adjoint bundle of $\mathscr{P}$, in the usual way by fixing some reference connection ω_0. Then the graded vector space of the covariant L_∞-algebra is

$$V^{\rm cov} := V_0^{\rm cov} \oplus V_1^{\rm cov} \oplus V_2^{\rm cov} \oplus V_3^{\rm cov}$$

where

$$\begin{aligned} V_0^{\rm cov} &= \Gamma(TM) \times \Omega^0\big(M, \mathscr{P} \times_{\rm ad} \mathfrak{so}(p,q)\big)\ , \\ V_1^{\rm cov} &= \Omega^1(M, \mathscr{V}) \times \Omega^1\big(M, \mathscr{P} \times_{\rm ad} \mathfrak{so}(p,q)\big)\ , \\ V_2^{\rm cov} &= \Omega^{d-1}\big(M, \textstyle\bigwedge^{d-1}\mathscr{V}\big) \times \Omega^{d-1}\big(M, \textstyle\bigwedge^{d-2}\mathscr{V}\big)\ , \\ V_3^{\rm cov} &= \Omega^1\big(M, \Omega^d(M)\big) \times \Omega^d\big(M, \textstyle\bigwedge^{d-2}\mathscr{V}\big)\ . \end{aligned}$$

We denote elements of these vector spaces with the same symbols as previously.

We will only write out the brackets $\ell_n^{\rm cov}$ which differ from those of the non-covariant formulation of Section 5.1. The only existing brackets from (5.2) which are modified are

$$\begin{aligned}
\ell_2^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\big) &= \big([\xi_1,\xi_2]\,,\,-[\rho_1,\rho_2]\big)\ ,\\
\ell_2^{\rm cov}\big((\xi,\rho)\,,\,(e,\omega)\big) &= \big(-\rho\cdot e+{\rm L}_\xi e\,,\,-[\rho,\omega]+\iota_\xi{\rm d}\omega\big)\ ,\\
\ell_2^{\rm cov}\big((\xi,\rho)\,,\,(E,\varOmega)\big) &= \big({\rm L}_\xi E-\rho\cdot E\,,\,{\rm d}\iota_\xi\varOmega-\rho\cdot\varOmega\big)\ ,\\
\ell_2^{\rm cov}\big((\xi,\rho)\,,\,(\mathcal{X},\mathcal{P})\big) &= \big({\rm L}_\xi\mathcal{X}\,,\,-\rho\cdot\mathcal{P}\big)\ ,\\
\ell_2^{\rm cov}\big((e,\omega)\,,\,(E,\varOmega)\big) &= \Big({\rm d}x^\mu\otimes{\rm Tr}\big(\iota_\mu{\rm d}e\curlywedge E+(-1)^{d-1}\,\iota_\mu{\rm d}\omega\curlywedge\varOmega-\iota_\mu e\curlywedge{\rm d}E\big)\,,\\
&\qquad\qquad \frac{d-1}{2}\,E\wedge e-(-1)^{d-1}\,\omega\wedge\varOmega\Big)\ .
\end{aligned}\tag{5.15}$$

There are also a number of new higher brackets that emerge. The non-trivial covariant 3-brackets are given by

$$\begin{aligned}
\ell_3^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\,,\,(e,\omega)\big) &= \big(0\,,\,-\iota_{\xi_1}\iota_{\xi_2}{\rm d}\omega\big)\ ,\\
\ell_3^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\,,\,(\mathcal{X},\mathcal{P})\big) &= \big(0\,,\,-{\rm d}\iota_{\xi_1}\iota_{\xi_2}\mathcal{P}\big)\ ,\\
\ell_3^{\rm cov}\big((\xi,\rho)\,,\,(e_1,\omega_1)\,,\,(e_2,\omega_2)\big) &= -\big(\iota_\xi\omega_1\cdot e_2+\iota_\xi\omega_2\cdot e_1\,,\,[\iota_\xi\omega_1,\omega_2]+[\iota_\xi\omega_2,\omega_1]\big)\ ,\\
\ell_3^{\rm cov}\big((\xi,\rho)\,,\,(E,\varOmega)\,,\,(e,\omega)\big) &= \Big(\iota_\xi\omega\cdot E\,,\,\omega\wedge\iota_\xi\varOmega+(-1)^{d-1}\,\frac{d-1}{2}\,\iota_\xi(E\wedge e)\Big)\ ,\\
\ell_3^{\rm cov}\big((\xi,\rho)\,,\,(e,\omega)\,,\,(\mathcal{X},\mathcal{P})\big) &= \big({\rm d}x^\mu\otimes{\rm Tr}(\iota_\mu\iota_\xi{\rm d}\omega\curlywedge\mathcal{P})\,,\,0\big)\ ,\\
\ell_3^{\rm cov}\big((E,\varOmega)\,,\,(e_1,\omega_1)\,,\,(e_2,\omega_2)\big) &= -\Big(\frac{d-1}{2}\,{\rm d}x^\mu\otimes{\rm Tr}\big(\iota_\mu\omega_1\curlywedge(E\wedge e_2)+\iota_\mu\omega_2\curlywedge(E\wedge e_1)\big)\\
&\qquad -(-1)^{d-1}\,{\rm d}x^\mu\otimes{\rm Tr}\big(\iota_\mu\omega_1\curlywedge(\omega_2\wedge\varOmega)+\iota_\mu\omega_2\curlywedge(\omega_1\wedge\varOmega)\big)\,,\,0\Big)
\end{aligned}\tag{5.16}$$

while the non-trivial covariant 4-brackets are given by

$$\begin{aligned}
\ell_4^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\,,\,(e_1,\omega_1)\,,\,(e_2,\omega_2)\big) &= \big(0\,,\,\iota_{\xi_1}\iota_{\xi_2}[\omega_1,\omega_2]\big)\ ,\\
\ell_4^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\,,\,(e,\omega)\,,\,(\mathcal{X},\mathcal{P})\big) &= \big(0\,,\,(-1)^{d-1}\,\omega\wedge\iota_{\xi_1}\iota_{\xi_2}\mathcal{P}\big)\ ,\\
\ell_4^{\rm cov}\big((\xi,\rho)\,,\,(e_1,\omega_1)\,,\,(e_2,\omega_2)\,,\,(\mathcal{X},\mathcal{P})\big) &= \big({\rm d}x^\mu\otimes{\rm Tr}(\iota_\mu\iota_\xi[\omega_1,\omega_2]\curlywedge\mathcal{P})\,,\,0\big)\ .
\end{aligned}\tag{5.17}$$

On all other fields and in all other degrees, the covariant brackets coincide with the brackets of Section 5.1: $\ell_n^{\rm cov}=\ell_n$ otherwise. The proof of the homotopy relations for these covariant brackets is discussed in Appendix A.2.

It is straightforward to check that the covariant brackets continue to encode all kinematical and dynamical information about Einstein–Cartan–Palatini gravity, now incorporating the covariant infinitesimal action of diffeomorphisms discussed previously. Strictly speaking, the brackets written above only make sense on local trivializations of the underlying vector bundles. However, the $\ell_n^{\rm cov}$-polynomial expressions of physical interest patch up to global objects, by covariance. For example, the gauge transformations of the dynamical fields are now given by

$$\delta^{\rm cov}_{(\xi,\rho)}(e,\omega)=\ell_1^{\rm cov}(\xi,\rho)+\ell_2^{\rm cov}\big((\xi,\rho)\,,\,(e,\omega)\big)-\frac12\,\ell_3^{\rm cov}\big((\xi,\rho)\,,\,(e,\omega)\,,\,(e,\omega)\big)\ \in\ V_1^{\rm cov}\ ,$$

where a non-trivial 3-bracket $\ell_3^{\rm cov}$ arises because (5.13) now involves third degree polynomial combinations. Similarly, one may read off the brackets from the polynomial expression for the Noether identities, which now includes a non-trivial 3-bracket $\ell_3^{\rm cov}$ by (5.14), so that

$$\mathsf{d}^{\rm cov}_{(e,\omega)}(\mathcal{F}_e,\mathcal{F}_\omega)=\ell_1^{\rm cov}(\mathcal{F}_e,\mathcal{F}_\omega)+\ell_2^{\rm cov}\big((\mathcal{F}_e,\mathcal{F}_\omega)\,,\,(e,\omega)\big)-\frac12\,\ell_3^{\rm cov}\big((\mathcal{F}_e,\mathcal{F}_\omega)\,,\,(e,\omega)\,,\,(e,\omega)\big)\ \in\ V_3^{\rm cov}\ .$$

However, it is not immediately obvious that these new brackets encode the expected covariance of the Euler–Lagrange derivatives, that is,

$$\delta^{\rm cov}_{(\xi,\rho)}(\mathcal{F}_e,\mathcal{F}_\omega) = \ell_2^{\rm cov}\big((\xi,\rho)\,,\,(\mathcal{F}_e,\mathcal{F}_\omega)\big) + \ell_3^{\rm cov}\big((\xi,\rho)\,,\,(\mathcal{F}_e,\mathcal{F}_\omega)\,,\,(e,\omega)\big)\ .$$

The part concerning local pseudo-orthogonal rotations is immediate, so that expanding the right-hand side for $\rho=0$ we confirm

$$\begin{aligned}
\big(\mathrm{L}_\xi\mathcal{F}_e\,,\,\mathrm{d}\iota_\xi\mathcal{F}_\omega\big) &+ \Big(\iota_\xi\omega\cdot\mathcal{F}_e\,,\,\omega\wedge\iota_\xi\mathcal{F}_\omega + (-1)^{d-1}\,\frac{d-1}{2}\,\iota_\xi(\mathcal{F}_e\wedge e)\Big)\\
&= \Big(\mathrm{L}^\omega_\xi\mathcal{F}_e\,,\,\mathrm{L}_\xi\mathcal{F}_\omega - \iota_\xi\mathrm{d}\mathcal{F}_\omega - \iota_\xi(\omega\wedge\mathcal{F}_\omega) + \iota_\xi\omega\cdot\mathcal{F}_\omega + (-1)^{d-1}\,\frac{d-1}{2}\,\iota_\xi(\mathcal{F}_e\wedge e)\Big)\\
&= \Big(\mathrm{L}^\omega_\xi\mathcal{F}_e\,,\,\mathrm{L}^\omega_\xi\mathcal{F}_\omega - \iota_\xi\Big(\mathrm{d}^\omega\mathcal{F}_\omega - (-1)^{d-1}\,\frac{d-1}{2}\,\mathcal{F}_e\wedge e\Big)\Big)\\
&= \big(\mathrm{L}^\omega_\xi\mathcal{F}_e\,,\,\mathrm{L}^\omega_\xi\mathcal{F}_\omega\big)\\
&= \delta^{\rm cov}_{(\xi,0)}(\mathcal{F}_e,\mathcal{F}_\omega)\ ,
\end{aligned}$$

where in the first equality we used $\mathrm{L}_\xi=\iota_\xi\circ\mathrm{d}+\mathrm{d}\circ\iota_\xi$ and $\iota_\xi(\omega\wedge\mathcal{F}_\omega)=\iota_\xi\omega\cdot\mathcal{F}_\omega-\omega\wedge\iota_\xi\mathcal{F}_\omega$, together with the definition of L^ω_ξ acting on the Euler–Lagrange derivatives which are forms valued in vector bundles associated to multivector representations of $\mathsf{SO}_+(p,q)$. In the second equality we used again the definition of L^ω_ξ together with d^ω, while in the fourth equality we used the Noether identity corresponding to invariance under local pseudo-orthogonal rotations. From this perspective, the input of the Noether identities is crucial. The naive bootstrap method, excluding the demand of cyclicity, would result in a simpler version of the brackets avoiding the use of the Noether identities. However, the resulting L_∞-algebra would not be cyclic with respect to the natural pairing introduced in Section 5.2: The requirement of cyclicity modifies the brackets via the application of the Noether identities.

Another new feature which appears here is in the closure of the gauge transformations. The covariant brackets also encode these, but now in the more general sense (2.14) where the bracket of the gauge algebra is field-dependent (but closure still holds off-shell):

$$\big[\delta^{\rm cov}_{(\xi_1,\rho_1)}\,,\,\delta^{\rm cov}_{(\xi_2,\rho_2)}\big](e,\omega) = \delta^{\rm cov}_{[\![(\xi_1,\rho_1),(\xi_2,\rho_2)]\!]^{\rm cov}_{(e,\omega)}}(e,\omega)\ ,$$

where

$$\begin{aligned}
[\![(\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)]\!]^{\rm cov}_{(e,\omega)} &= -\,\ell_2^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\big)\\
&\qquad - \ell_3^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\,,\,(e,\omega)\big) + \frac12\,\ell_4^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\,,\,(e,\omega)\,,\,(e,\omega)\big)\\
&= \big(-[\xi_1,\xi_2]\,,\,[\rho_1,\rho_2]+\iota_{\xi_1}\iota_{\xi_2}R\big)\ .
\end{aligned}\tag{5.18}$$

This encodes directly the possible non-integrability of the horizontal lifting corresponding to each connection (5.12). In particular, this means that the space of fields $V_1^{\rm cov}$ does *not* form a module over the Lie algebra of gauge transformations on $V_0^{\rm cov}$, in marked contrast with the non-covariant approach, see (5.4). This formula is also noted in [24], albeit in the dual and shifted picture in which the Lie derivative is viewed as an odd operator, where it is shown that the usual Cartan identity

$$\mathrm{L}_{[\xi_1,\xi_2]} = \mathrm{L}_{\xi_1}\circ\mathrm{L}_{\xi_2} - \mathrm{L}_{\xi_2}\circ\mathrm{L}_{\xi_1}\tag{5.19}$$

is violated by the covariant Lie derivative L^ω_ξ via a term involving the action of the contracted curvature $\iota_{\xi_1}\iota_{\xi_2}R\in\Omega^0\big(M,\mathscr{P}\times_{\rm ad}\mathfrak{so}(p,q)\big)$, as in (5.18).

5.4 Cyclic L_∞-isomorphism

In this section we have introduced two L_∞-algebra formulations of ECP gravity, one local and the other capturing the requisite covariance properties for non-trivial spacetimes M. We would now like to show that these two formulations are physically equivalent locally, in the sense discussed in Section 2.3. In fact, we exhibit a stronger result: In the case where the underlying manifold M is parallelizable, the two theories are equivalent in the sense that their underlying L_∞-algebras are isomorphic. Indeed, all vector bundles in question are then trivial, and so the underlying vector spaces of the covariant and non-covariant formulations are identical, $V^{\rm cov} = V$. The L_∞-morphism we present here has been constructed partly via the help of dualisation from the symplectomorphism demonstrated in [24] for d=4. In fact since the map does not interact with dynamics, it has formally the same form in any dimension.

Let $\{\psi_n^{\rm cov}\}$ be the collection of multilinear graded antisymmetric maps

$$\psi_n^{\rm cov} : \bigwedge^n V^{\rm cov} \longrightarrow V \ ,$$

of degree $|\psi_n^{\rm cov}| = 1 - n$ for $n \geq 1$, defined as follows: $\psi_1^{\rm cov} : V^{\rm cov} \to V$ is the identity map

$$\psi_1^{\rm cov}(v) = v$$

for all $v \in V^{\rm cov}$, the map $\psi_2^{\rm cov} : \bigwedge^2 V^{\rm cov} \to V$ has only non-trivial components given by

$$\begin{aligned}
\psi_2^{\rm cov}\big((\xi,\rho)\,,\,(e,\omega)\big) &= \big(0\,,\,-\iota_\xi\omega\big) \ \in \ V_0 \ , \\
\psi_2^{\rm cov}\big((\xi,\rho)\,,\,(\mathcal{X},\mathcal{P})\big) &= \big(0\,,\,-(-1)^{d-1}\,\iota_\xi\mathcal{P}\big) \ \in \ V_2 \ , \\
\psi_2^{\rm cov}\big((e,\omega)\,,\,(\mathcal{X},\mathcal{P})\big) &= \big(-\mathrm{d}x^\mu \otimes \mathrm{Tr}(\iota_\mu\omega \curlywedge \mathcal{P})\,,\,0\big) \ \in \ V_3 \ ,
\end{aligned}$$

while $\psi_n^{\rm cov} = 0$ for all $n \geq 3$. Then $\{\psi_n^{\rm cov}\}$ is a cyclic L_∞-isomorphism between the cyclic L_∞-algebras $\big(V^{\rm cov}, \{\ell_n^{\rm cov}\}, \langle -,-\rangle\big)$ and $\big(V, \{\ell_n\}, \langle -,-\rangle\big)$. One easily verifies the Seiberg–Witten maps from (2.16)–(2.19) in this instance with

$$(e,\omega)^{\rm cov} = (e,\omega) \ , \quad (\mathcal{F}_e,\mathcal{F}_\omega)^{\rm cov} = (\mathcal{F}_e,\mathcal{F}_\omega) \qquad \text{and} \qquad (\xi,\rho)^{\rm cov} = (\xi,\rho-\iota_\xi\omega) \ ,$$

so that $\delta^{\rm cov}_{(\xi,\rho)}(e,\omega) = \delta_{(\xi,\rho-\iota_\xi\omega)}(e,\omega)$ and $\delta^{\rm cov}_{(\xi,\rho)}(\mathcal{F}_e,\mathcal{F}_\omega) = \delta_{(\xi,\rho-\iota_\xi\omega)}(\mathcal{F}_e,\mathcal{F}_\omega)$ with the gauge algebra mapping as

$$\begin{aligned}
[\![(\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)]\!]^{\rm cov}_{(e,\omega)} = &-\ell_2\big((\xi_1,\rho_1-\iota_{\xi_1}\omega)\,,\,(\xi_2,\rho_2-\iota_{\xi_2}\omega)\big) \\
&+ \big(\xi_2-\xi_1\,,\,\rho_2-\rho_1+\iota_{\xi_1}\delta_{(\xi_2,\rho_2-\iota_{\xi_2}\omega)}\omega - \iota_{\xi_2}\delta_{(\xi_1,\rho_1-\iota_{\xi_1}\omega)}\omega\big) \ .
\end{aligned}$$

Despite their simplicity, showing that the maps $\{\psi_n^{\rm cov}\}$ satisfy the required relations (2.5) of an L_∞-morphism is, like the proof of the homotopy relations, a tedious calculation which largely does not depend on the spacetime dimension d. We give the proof for the case $d = 3$ in Appendix A.3; the proof is similar for $d \geq 4$.

Because $\{\psi_n^{\rm cov}\}$ is an L_∞-morphism, since $\psi_1^{\rm cov}$ is the identity it follows that $\{\psi_n^{\rm cov}\}$ is an L_∞-isomorphism. To check cyclicity of the map, since the cyclic pairing is the same on both L_∞-algebras of the gravity theory, it follows immediately that

$$\langle\psi_1^{\rm cov}(v_1),\psi_1^{\rm cov}(v_2)\rangle = \langle v_1,v_2\rangle$$

for all $v_1,v_2 \in V^{\rm cov}$, again because $\psi_1^{\rm cov}$ is the identity. Furthermore, it is a straightforward degree-wise calculation to check that

$$(-1)^{|v_1|}\,\langle\psi_1^{\rm cov}(v_1),\psi_2^{\rm cov}(v_2,v_3)\rangle - \langle\psi_2^{\rm cov}(v_1,v_2),\psi_1^{\rm cov}(v_3)\rangle = 0$$

and

$$\langle\psi_2^{\rm cov}(v_1,v_2),\psi_2^{\rm cov}(v_3,v_4)\rangle = 0 \ ,$$

for all $v_1,v_2,v_3,v_4 \in V^{\rm cov}$. The remaining cyclicity relations are all trivial.

6 BV–BRST formalism for Einstein–Cartan–Palatini gravity

The duality between differential graded commutative algebras and L_∞-algebras of finite type discussed in Section 2.2 converts the BV complex of a classical field theory into an L_∞-algebra as described in [15], and *vice versa*. In this section we shall explicitly demonstrate this fact in the case of the non-covariant ECP formalism, after reviewing the BRST complex of ECP gravity [52–54] and its augmented BV-BRST version [24], following the conventions of [17], where one may find a detailed introduction to the subject. The covariant BV-BRST complex of [24] proceeds analogously and is indeed dual to the covariant L_∞-algebra presented in the last section.

6.1 BRST complex

The BRST complex for ECP gravity in d dimensions is obtained as the Chevalley–Eilenberg resolution for the quotient of the space of fields (3.5) by the gauge algebra (3.6). It has underlying vector space

$$\mathscr{F}_{\text{BRST}} = \mathscr{F}_{\text{BRST}\,0} \ \oplus\ \mathscr{F}_{\text{BRST}\,-1} \ ,$$

where

$$\mathscr{F}_{\text{BRST}\,0} = \Omega^1(M,\mathbb{R}^{p,q}) \times \Omega^1\big(M,\mathfrak{so}(p,q)\big) \ ,$$
$$\mathscr{F}_{\text{BRST}\,-1} = \Gamma[1](TM) \times \Omega^0[1]\big(M,\mathfrak{so}(p,q)\big) \ ,$$

so that the dynamical fields are elements $(e,\omega) \in \mathscr{F}_{\text{BRST}\,0}$ and the gauge parameters are elements[23] $(\xi,\rho) \in \mathscr{F}_{\text{BRST}\,-1}$ with $e = e^a\,\mathtt{E}_a$, $\omega = \omega^{ab}\,\mathtt{E}_{ba}$ and $\rho = \rho^{ab}\,\mathtt{E}_{ba}$. In the language of the BRST formalism, the elements of the odd degree spaces of gauge parameters, which define sections of a distribution $\mathscr{D} \subset T\mathscr{F}_{\text{BRST}\,0}$ with a degree shift of 1, are called ghosts.

On a local chart for M with coordinates $x = (x^\mu)$, the fields are expanded in holonomic bases as $e = e^a_\mu(x)\,\mathrm{d}x^\mu\,\mathtt{E}_a$ and $\xi = \xi^\mu(x)\,\partial_\mu$, where $\partial_\mu = \frac{\partial}{\partial x^\mu}$, and similarly for the rest of the fields. Abusing notation slightly, we shall consider the components e^a_μ as elements of the dual space $\mathscr{F}^\star_{\text{BRST}}$, thus viewing $e^a_\mu(x)$ as coordinate functions on the infinite-dimensional vector space $\Omega^1(M,\mathbb{R}^{p,q})$ via the evaluation map

$${e'}^a_{\mu\,|x} : \Omega^1(M,\mathbb{R}^{p,q}) \longrightarrow \mathbb{R} \ , \quad e \longmapsto e^a_\mu(x) \ ,$$

and similarly for the rest of the fields. Abusing notation slightly, we will sometimes drop the primes in the following.

The BRST differential Q_{BRST} should act on a suitable space of functionals of the field complex, which we denote by $\mathcal{O}(\mathscr{F}_{\text{BRST}})$. The precise definition of this space will not be of concern to us, and it is often different depending on the context and goals. For our purposes, the following naive description will suffice: Consider $\mathscr{F}^\star_{\text{BRST}} := \mathsf{Hom}(\mathscr{F}_{\text{BRST}},\mathbb{R})$, the space of (continuous) $\mathbb{R}$-linear functionals on $\mathscr{F}_{\text{BRST}}$; note that these are *not* sections of the dual bundles. This space includes the coordinate maps ${e'}^a_{\mu\,|x}$ above, as well as maps factoring through the jet bundles, such as $\partial_\nu {e'}^a_{\mu\,|x}$, which extract the values of derivatives of the fields at a point in a specified coordinate chart of the underlying manifold M. We shall take $\mathcal{O}(\mathscr{F}_{\text{BRST}}) := \bigodot^\bullet_{\mathbb{R}} \mathscr{F}^\star_{\text{BRST}}$, the symmetric tensor algebra over $\mathbb{R}$ of the dual of $\mathscr{F}_{\text{BRST}}$, as the space of polynomial functionals on the field complex. The usual

[23] Strictly speaking these should be denoted as $({}^s\xi, {}^s\rho)$, where (ξ,ρ) are the gauge parameters introduced in Section 3 and s is the suspension isomorphism in (6.2) below, but we do not indicate s explicitly in order to streamline our formulas in the following.

subtleties regarding the topology of $\mathcal{F}^\star_{\rm BRST}$ and in which category the tensor product is taken are treated in detail in [36, 51]. For our purposes, it will be safe to treat this tensor product formally as the algebraic tensor product.

Then $Q_{\rm BRST}$ is a degree 1 derivation $Q_{\rm BRST} : \odot^\bullet_{\mathbb{R}}\mathscr{F}^\star_{\rm BRST} \to \odot^\bullet_{\mathbb{R}}\mathscr{F}^\star_{\rm BRST}$ such that $Q^2_{\rm BRST} = 0$. By virtue of being a derivation, $Q_{\rm BRST}$ is completely determined by its action on the basis elements of $\mathscr{F}^\star_{\rm BRST}$. For Einstein–Cartan–Palatini gravity, this takes the form [52–54]

$$
\begin{aligned}
Q_{\rm BRST}e^a_\mu &= {\rm L}_\xi e^a_\mu - (\rho\cdot e)^a_\mu = (\xi^\nu\odot\partial_\nu e^a_\mu + e^a_\nu\odot\partial_\mu\xi^\nu) - \rho^a{}_b\odot e^b_\mu\ ,\\
Q_{\rm BRST}\omega^{ab}_\mu &= {\rm L}_\xi\omega^{ab}_\mu + {\rm d}^\omega\rho^{ab}{}_\mu = (\xi^\nu\odot\partial_\nu\omega^{ab}_\mu + \omega^{ab}_\nu\odot\partial_\mu\xi^\nu) + \partial_\mu\rho^{ab}\odot 1 + \omega^{ac}_\mu\odot\rho_c{}^b - \rho^a{}_c\odot\omega^{cb}_\mu\ ,\\
Q_{\rm BRST}\xi^\mu &= \tfrac12\,[\xi,\xi]^\mu = \xi^\nu\odot\partial_\nu\xi^\mu\ ,\\
Q_{\rm BRST}\rho^{ab} &= {\rm L}_\xi\rho^{ab} - \tfrac12\,[\rho,\rho]^{ab} = \xi^\nu\odot\partial_\nu\rho^{ab} - \rho^a{}_c\odot\rho^{cb}\ .
\end{aligned}
\tag{6.1}
$$

The BRST operator $Q_{\rm BRST}$ encodes the symmetries of ECP theory, that is, physical (gauge-invariant) states modulo gauge transformations are classes in the degree 0 cohomology of the BRST complex; in particular, the action functional $S_{\rm ECP}$ is a cocycle in degree 0: $Q_{\rm BRST}S_{\rm ECP} = 0$. In a non-holonomic basis of vector fields $\{p_\alpha\}$ for $\Gamma(TM)$ with Lie brackets $[p_\alpha, p_\beta] = f^\gamma{}_{\alpha\beta}\,p_\gamma$, the third BRST transformation takes the form

$$
Q_{\rm BRST}\xi^\alpha = \tfrac12\,[\xi,\xi]^\alpha = \tfrac12\,f^\alpha{}_{\beta\gamma}\,\xi^\beta\odot\xi^\gamma + \xi^\beta\odot p_\beta(\xi^\alpha)\ ,
$$

which illustrates the similarity to the Chevalley–Eilenberg differential of a finite-dimensional Lie algebra. However, this is *not* the Chevalley–Eilenberg dual of $\mathscr{F}_{\rm BRST}$ as a $C^\infty(M)$-module; for instance, restricting to $\big(\Gamma(TM), [-,-]\big)$, this would give the de Rham complex $\big(\Omega^\bullet(M), {\rm d}\big)$ instead.

The BRST differential contains all of the kinematical gauge structure of ECP gravity. For this, we note that the differential $Q_{\rm BRST}$ dualizes to a codifferential

$$
{\rm D}_{\rm BRST} = Q^\star_{\rm BRST} : \odot^\bullet\mathscr{F}_{\rm BRST} \longrightarrow \odot^\bullet\mathscr{F}_{\rm BRST}\ ,
$$

which may be decomposed as

$$
{\rm pr}_{\mathscr{F}_{\rm BRST}}\circ{\rm D}_{\rm BRST} = \sum_{n=1}^\infty {\rm D}_{{\rm BRST}\,n}
$$

where the components are maps ${\rm D}_{{\rm BRST}\,n} : \odot^n\mathscr{F}_{\rm BRST} \to \mathscr{F}_{\rm BRST}$. Introduce the suspension isomorphism $s : \mathscr{F}_{\rm BRST}[-1] \to \mathscr{F}_{\rm BRST}$, which induces an isomorphism of graded algebras given by

$$
\begin{aligned}
&s^{\otimes n} : \textstyle\bigwedge^n\mathscr{F}_{\rm BRST}[-1] \longrightarrow \odot^n\mathscr{F}_{\rm BRST}\ ,\\
&{}^{s^{-1}}v_1\wedge\cdots\wedge{}^{s^{-1}}v_n \longmapsto (-1)^{\sum_{j=1}^{n-1}(n-j)\,|{}^{s^{-1}}v_j|}\,v_1\odot\cdots\odot v_n\ .
\end{aligned}
\tag{6.2}
$$

Then the graded antisymmetric brackets of the kinematical part of the L_∞-algebra that we defined in Section 5.1 are given exactly by

$$
\ell_n := s^{-1}\circ{\rm D}_{{\rm BRST}\,n}\circ s^{\otimes n} : \textstyle\bigwedge^n\mathscr{F}_{\rm BRST}[-1] \longrightarrow \mathscr{F}_{\rm BRST}[-1]\ ,
$$

and nilpotence $Q^2_{\rm BRST} = 0$ translates to the homotopy relations for the brackets [17]. Dualizing back and forth in this infinite-dimensional case is a delicate issue; however, the formal dualization below make sense because one may interpret the brackets ℓ_n as maps between respective jet bundles, and then dualize pointwise.

We will now indicate how to explicitly calculate these brackets. For a diffeomorphism $\xi \in \mathscr{F}_{\text{BRST}\,-1}$ and a coframe field $e \in \mathscr{F}_{\text{BRST}\,0}$, using the natural duality pairing $\langle - | - \rangle$ between $\bigodot^\bullet \mathscr{F}_{\text{BRST}}$ and $\bigodot^\bullet_{\mathbb{R}} \mathscr{F}^\star_{\text{BRST}}$ we get

$$
\begin{aligned}
\langle Q_{\text{BRST}} e'^{\,a}_\mu | \xi \odot e \rangle &= \langle \xi'^{\,\nu} \odot \partial_\nu e'^{\,a}_\mu + e'^{\,a}_\nu \odot \partial_\mu \xi'^{\,\nu} | \xi \odot e \rangle \\
&= (-1)^{|e'|\,|\xi'|} \, (\xi^\nu \, \partial_\nu e^a_\mu + e^a_\nu \, \partial_\mu \xi^\nu) \\
&= \langle e'^{\,a}_\mu | \mathrm{L}_\xi e \rangle \\
&=: \langle e'^{\,a}_\mu | (-1)^{|Q_{\text{BRST}}|\,|e'|} \, \mathrm{D}_{\text{BRST}\,2}(\xi \odot e) \rangle \\
&= \langle e'^{\,a}_\mu | \mathrm{D}_{\text{BRST}\,2}(\xi \odot e) \rangle
\end{aligned}
$$

where we used $|Q_{\text{BRST}}| = 1$, $|e'| = 0$ and $|\xi| = -1$. Hence $\mathrm{D}_{\text{BRST}\,2}(\xi \odot e) = \mathrm{L}_\xi e$, and so[24]

$$
\begin{aligned}
\ell_2({}^{s^{-1}}\xi \wedge {}^{s^{-1}}e) &= s^{-1} \circ \mathrm{D}_{\text{BRST}\,2} \circ (s \otimes s)({}^{s^{-1}}\xi \wedge {}^{s^{-1}}e) \\
&= (-1)^{|{}^{s^{-1}}\xi|\,|s|} \, s^{-1} \circ \mathrm{D}_{\text{BRST}\,2}(\xi \odot e) \\
&= s^{-1} \circ \mathrm{D}_{\text{BRST}\,2}(\xi \odot e) \\
&= \mathrm{L}_{{}^{s^{-1}}\xi} {}^{s^{-1}}e \ ,
\end{aligned}
$$

which agrees with (5.2). Similarly, for a local pseudo-orthogonal rotation $\rho \in \mathscr{F}_{\text{BRST}\,-1}$ we calculate

$$
\begin{aligned}
\langle Q_{\text{BRST}} e'^{\,a}_\mu | \rho \odot e \rangle &= \langle -\rho'^{\,a}{}_b \odot e'^{\,b}_\mu | \rho \odot e \rangle \\
&= -(-1)^{|\rho'|\,|e'|} \, \rho^a{}_b \, e^b_\mu \\
&= -\langle e'^{\,a}_\mu | \rho \cdot e \rangle \\
&=: \langle e'^{\,a}_\mu | -(-1)^{|Q_{\text{BRST}}|\,|e'|} \, \mathrm{D}_{\text{BRST}\,2}(\rho \odot e) \rangle \ .
\end{aligned}
$$

Hence $\mathrm{D}_{\text{BRST}\,2}(\rho \odot e) = -\rho \cdot e$, and so

$$
\begin{aligned}
\ell_2({}^{s^{-1}}\rho \wedge {}^{s^{-1}}e) &= s^{-1} \circ \mathrm{D}_{\text{BRST}\,2} \circ (s \otimes s)({}^{s^{-1}}\rho \wedge {}^{s^{-1}}e) \\
&= (-1)^{|{}^{s^{-1}}\rho|\,|s|} \, s^{-1} \circ \mathrm{D}_{\text{BRST}\,2}(\rho \odot e) \\
&= s^{-1} \circ \mathrm{D}_{\text{BRST}\,2}(\rho \odot e) \\
&= -{}^{s^{-1}}\rho \cdot {}^{s^{-1}}e \ ,
\end{aligned}
$$

which also agrees with the corresponding brackets in (5.2). In a similar fashion one dualizes the rest of Q_{BRST} and recovers all of the brackets containing the kinematical gauge structure of the field theory, which in this case are guaranteed to satisfy the corresponding homotopy relations, since $Q^2_{\text{BRST}} = 0$.

6.2 BV–BRST complex

We now need to augment the kinematical gauge structure provided in the BRST formalism of Section 6.1 by the dynamical data comprising the field equations and the Noether identities. Starting

[24] Yet another exterior product appears here: A wedge product inside a bracket denotes the exterior product of $\bigwedge^\bullet \mathscr{F}_{\text{BRST}}[-1]$.

from the classical BRST complex $\mathscr{F}_{\rm BRST}$ from Section 6.1, the BV complex provides the Koszul–Tate resolution of its degree 0 cohomology (gauge equivalence classes of fields) modulo the ideal of Euler–Lagrange derivatives. It is defined as the functionals $\mathcal{O}(\mathscr{F}_{\rm BV})$ on its shifted cotangent bundle:

$$\mathscr{F}_{\rm BV} := T^*[-1]\mathscr{F}_{\rm BRST} \ . \tag{6.3}$$

This takes the form

$$\mathscr{F}_{\rm BV} = \mathscr{F}_{{\rm BV}\,-1} \ \oplus \ \mathscr{F}_{{\rm BV}\,0} \ \oplus \ \mathscr{F}_{{\rm BV}\,1} \ \oplus \ \mathscr{F}_{{\rm BV}\,2} \ , \tag{6.4}$$

with

$$\begin{aligned}
\mathscr{F}_{{\rm BV}\,-1} &= \Gamma[1](TM) \times \Omega^0[1]\big(M,\mathfrak{so}(p,q)\big) \ , \\
\mathscr{F}_{{\rm BV}\,0} &= \Omega^1(M,\mathbb{R}^{p,q}) \times \Omega^1\big(M,\mathfrak{so}(p,q)\big) \ , \\
\mathscr{F}_{{\rm BV}\,1} &= \Omega^{d-1}[-1]\big(M,\textstyle\bigwedge^{d-1}(\mathbb{R}^{p,q})\big) \times \Omega^{d-1}[-1]\big(M,\textstyle\bigwedge^{d-2}(\mathbb{R}^{p,q})\big) \ , \\
\mathscr{F}_{{\rm BV}\,2} &= \Omega^1[-2]\big(M,\Omega^d(M)\big) \times \Omega^d[-2]\big(M,\textstyle\bigwedge^{d-2}(\mathbb{R}^{p,q})\big) \ ,
\end{aligned}$$

where elements of the degree 1 spaces are called antifields, which we denote by $(e^\dagger,\omega^\dagger)$, while elements of the degree 2 spaces are called antighosts,[25] denoted by $(\xi^\dagger,\rho^\dagger)$. The antifields and antighosts are transversal sections to the gauge orbits in the space of fields and ghosts (odd gauge parameters) respectively, and they may be paired with fields and ghosts using the pairing which defines the action functional $S_{\rm ECP}$ to give an $\mathbb{R}$-valued d-form on M.

Explicitly, the pairing in the Einstein–Cartan–Palatini action functional may be regarded as a non-degenerate bilinear form

$${\rm Tr}(-\curlywedge-) : \Omega^{d-k}\big(M,\textstyle\bigwedge^{d-k}(\mathbb{R}^{p,q})\big) \otimes \Omega^k\big(M,\textstyle\bigwedge^k(\mathbb{R}^{p,q})\big) \longrightarrow \Omega^d(M)$$

which defines the BV pairing $\langle-,-\rangle_{\rm BV}$ and dualizes the fields and ghosts to antifields and antighosts through the assignments

$$\begin{aligned}
e\in\Omega^1(M,\mathbb{R}^{p,q}) \ &\Longrightarrow \ e^\dagger\in\Omega^{d-1}\big(M,\textstyle\bigwedge^{d-1}(\mathbb{R}^{p,q})\big) \ , \\
\omega\in\Omega^1\big(M,\textstyle\bigwedge^2(\mathbb{R}^{p,q})\big) \ &\Longrightarrow \ \omega^\dagger\in\Omega^{d-1}\big(M,\textstyle\bigwedge^{d-2}(\mathbb{R}^{p,q})\big) \ , \\
\rho\in\Omega^0\big(M,\textstyle\bigwedge^2(\mathbb{R}^{p,q})\big) \ &\Longrightarrow \ \rho^\dagger\in\Omega^{d}\big(M,\textstyle\bigwedge^{d-2}(\mathbb{R}^{p,q})\big) \ , \\
\xi\in\Gamma(TM) \ &\Longrightarrow \ \xi^\dagger\in\Omega^1\big(M,\Omega^d(M)\big) \ ,
\end{aligned}$$

where the final duality is defined by $\iota_\xi\xi^\dagger\in\Omega^d(M)$. Here the spaces of antifields and antighosts correspond exactly to the spaces of Euler–Lagrange derivatives and of Noether identities; indeed, the graded vector space (6.4) is just the vector space (5.1) underlying our L_∞-algebra with degrees shifted by 1:

$$\mathscr{F}_{\rm BV} = V[1] \ ,$$

because of the degree shift by -1 in the cotangent bundle (6.3).

[25] These are not the antighost fields which are sometimes introduced in the BRST formalism, but this terminology is convenient in the present context.

The space $\mathscr{F}_{\rm BV}$, being a cotangent bundle, further has the structure of a graded (infinite-dimensional) (-1)-symplectic manifold, where the canonical symplectic two-form is given by

$$\boldsymbol{\omega}_{\rm BV} := \int_M \operatorname{Tr}\big(\delta e \curlywedge \delta e^\dagger + \delta\omega \curlywedge \delta\omega^\dagger - \delta\rho \curlywedge \delta\rho^\dagger\big) - \int_M \iota_{\delta\xi}\delta\xi^\dagger \ . \tag{6.5}$$

As expected, this is the shifted (and dual) version of the cyclic pairing 5.5. It induces a graded Poisson bracket $\{-,-\}_{\rm BV}$ of smooth functions on $\mathscr{F}_{\rm BV}$, that we consider to be the space $\odot^\bullet_{\mathbb{R}}\mathscr{F}^\star_{\rm BV}$ as in Section 6.1, which is called the antibracket. In this sense, fields and their antifield partners may be regarded as canonically conjugate variables.

We would now like to define the BV extension of the action functional $S_{\rm ECP}$ to a local action functional $S_{\rm BV}$ on $\mathscr{F}_{\rm BV}$ of degree 0 that satisfies the classical master equation

$$\{S_{\rm BV}, S_{\rm BV}\}_{\rm BV} = 0 \ . \tag{6.6}$$

This is always possible for BV complexes which are built out of underlying BRST complexes [55], and it automatically extends the BRST differential $Q_{\rm BRST}$ on $\mathscr{F}_{\rm BRST}$ to the BV differential $Q_{\rm BV}$ on $\mathscr{F}_{\rm BV}$ as the cotangent lift

$$Q_{\rm BV}F := \{S_{\rm BV}, F\}_{\rm BV} = -\int_M \sum_{A\in\{e,\omega,\rho,\xi\}} \Big(\Big\langle \frac{\delta S_{\rm BV}}{\delta A}, \frac{\delta F}{\delta A^\dagger}\Big\rangle^\star_{\rm BV} + \Big\langle \frac{\delta S_{\rm BV}}{\delta A^\dagger}, \frac{\delta F}{\delta A}\Big\rangle^\star_{\rm BV}\Big) \ ,$$

where $\langle -,-\rangle^\star_{\rm BV}$ denotes the dual BV pairing and $\frac{\delta}{\delta e} := \frac{\delta}{\delta e^a_\mu}\otimes \mathtt{F}^a\,\partial_\mu$ with $\mathtt{F}^a$ dual to $\mathtt{E}_a$ and ∂_μ dual to $\mathrm{d}x^\mu$, and similarly for the other fields. The classical master equation (6.6) together with the graded Jacobi identity for the antibracket guarantee that $Q^2_{\rm BV} = 0$.

A solution to (6.6) is given by the BV pairing as [55]

$$S_{\rm BV} = S_{\rm ECP} + \int_M \operatorname{Tr}\big(Q_{\rm BRST}e \curlywedge e^\dagger + Q_{\rm BRST}\omega \curlywedge \omega^\dagger - Q_{\rm BRST}\rho \curlywedge \rho^\dagger\big) - \int_M \iota_{Q_{\rm BRST}\xi}\xi^\dagger \ , \tag{6.7}$$

where $Q_{\rm BRST}e := Q_{\rm BRST}e^a_\mu \otimes \mathtt{E}_a\,\mathrm{d}x^\mu \in \odot^\bullet_{\mathbb{R}}\mathscr{F}^\star_{\rm BRST}\otimes\mathscr{F}_{\rm BRST}$, and similarly for the other fields. This defines the *minimal* part of the BV extension of the Einstein–Cartan–Palatini theory; the non-minimal part of the action functional is trivial from our classical perspective as it involves products of auxiliary fields and antighosts, and so is the $Q_{\rm BV}$-exact variation of a gauge fixing fermion. Explicitly, the minimal BV extension of the action functional for d-dimensional gravity reads

$$\begin{aligned} S_{\rm BV} = \int_M \operatorname{Tr}\Big(&\frac{1}{d-2}\,e^{d-2}\curlywedge R + \frac{1}{d}\,\Lambda\,e^d + \big(\mathrm{L}_\xi e - \rho\cdot e\big)\curlywedge e^\dagger + \big(\mathrm{L}_\xi\omega + \mathrm{d}^\omega\rho\big)\curlywedge\omega^\dagger\Big) \\ &- \frac{1}{2}\int_M \operatorname{Tr}\big((2\,\mathrm{L}_\xi\rho - [\rho,\rho])\curlywedge\rho^\dagger\big) - \frac{1}{2}\int_M \iota_{[\xi,\xi]}\xi^\dagger \ . \end{aligned}$$

From the form of the BV action functional (6.7), it immediately follows that the pullback of the BV differential to the natural Lagrangian submanifold of $\mathscr{F}_{\rm BV}$ provided by the zero section is given by

$$Q_{{\rm BV}|_{\mathscr{F}_{\rm BRST}}} = Q_{\rm BRST} \ ,$$

and so this part of the BV differential $Q_{\rm BV}$ includes the kinematical gauge sector of our L_∞-algebra via dualization. The BV transformations of the antifields $Q_{\rm BV}e^\dagger$ and $Q_{\rm BV}\omega^\dagger$ incorporate the dynamical brackets of our L_∞-algebra, while $Q_{\rm BV}\rho^\dagger$ and $Q_{\rm BV}\xi^\dagger$ encode brackets corresponding to the Noether identities and the action of the gauge parameters on the space of Noether identities [17, 38, 56]. This, together with the fact that $Q^2_{\rm BV} = 0$, guarantee that the homotopy relations are satisfied in any dimension $d \geq 3$. The canonical symplectic two-form (6.5) corresponds to the cyclic pairing of the L_∞-algebra introduced in Section 5.2, with cyclicity being equivalent to the $Q_{\rm BV}$-invariance of $\boldsymbol{\omega}_{\rm BV}$, and the sign difference comes from the degree shifting. Then cyclic L_∞-morphisms correspond to cohomomorphisms of $\mathscr{F}_{\rm BV}$, in the sense discussed in Section 2.2, whose duals preserve the symplectic structures. In particular, cyclic quasi-isomorphisms result into equivalent field theories, thus in our case relating equivalent gravity theories.

After some tedious but straightforward calculation, one may arrive at

$$
\begin{aligned}
Q_{\rm BV}e^{\dagger\,a_1\cdots a_{d-1}}_{\ \mu_1\cdots\mu_{d-1}} =&- \ e^{[a_1}_{[\mu_1}\cdots e^{a_{d-3}}_{\mu_{d-3}}\,R^{a_{d-2}a_{d-1}]}_{\mu_{d-2}\mu_{d-1}]} - \Lambda\, e^{[a_1}_{[\mu_1}\cdots e^{a_{d-1}]}_{\mu_{d-1}]} \\
&+ d\,\big(e^{\dagger\,a_1\cdots a_{d-1}}_{\ [\mu_1\cdots\mu_{d-1}}\,\partial_{\sigma]}\xi^\sigma - e^{\dagger\,[a_1\cdots a_{d-1}}_{\ \mu_1\cdots\mu_{d-1}}\,\rho^{b]}{}_b\big) + \partial_\sigma\big(\xi^\sigma\, e^{\dagger\,a_1\cdots a_{d-1}}_{\ \mu_1\cdots\mu_{d-1}}\big)\ , \\[2mm]
Q_{\rm BV}\omega^{\dagger\,a_1\cdots a_{d-2}}_{\ \mu_1\cdots\mu_{d-1}} =&- \ e^{[a_1}_{[\mu_1}\cdots e^{a_{d-3}}_{\mu_{d-3}}\,T^{a_{d-2}]}_{\ \mu_{d-2}\mu_{d-1}]} \\
&+ (d-1)\,\rho^{[b}{}_b\,\omega^{\dagger\,a_1\cdots a_{d-2}]}_{\ \mu_1\cdots\mu_{d-1}} + d\,\omega^{\dagger\,a_1\cdots a_{d-1}}_{\ [\mu_1\cdots\mu_{d-1}}\,\partial_{\sigma]}\xi^\sigma + \partial_\sigma\big(\xi^\sigma\, e^{\dagger\,a_1\cdots a_{d-1}}_{\ \mu_1\cdots\mu_{d-1}}\big)\ , \\[2mm]
Q_{\rm BV}\rho^{\dagger\,a_1\cdots a_{d-2}}_{\ \mu_1\cdots\mu_d} =&- \ \frac{d-1}{2}\,e_{b[\mu_1}\,e^{\dagger\,ba_1\cdots a_{d-2}}_{\ \mu_2\cdots\mu_d]} + (-1)^{d-1}\,\omega^{a_1}{}_{b[\mu_1}\,\omega^{\dagger\,ba_2\cdots a_{d-2}}_{\ \mu_2\cdots\mu_d]} + \partial_{[\mu_1}\omega^{\dagger\,a_1\cdots a_{d-2}}_{\ \mu_2\cdots\mu_d]} \\
&- \rho^{a_1}{}_b\,\rho^{\dagger\,ba_2\cdots a_{d-2}}_{\ \mu_1\cdots\mu_d} + \partial_\sigma\big(\xi^\sigma\,\rho^{\dagger\,a_1\cdots a_{d-2}}_{\ \mu_1\cdots\mu_d}\big)\ , \\[2mm]
Q_{\rm BV}\xi^\dagger_{\nu\mu_1\cdots\mu_d} =&- \varepsilon_{a_1\cdots a_d}\,\Big(\partial_\nu e^{a_1}_{[\mu_1}\,e^{\dagger\,a_2\cdots a_d}_{\ \mu_2\cdots\mu_d]} - \partial_{[\mu_1}e^{a_1}_{|\nu|}\,e^{\dagger\,a_2\cdots a_d}_{\ \mu_2\cdots\mu_d]} - e^{a_1}_{|\nu|}\,\partial_{[\mu_1}e^{\dagger\,a_2\cdots a_d}_{\ \mu_2\cdots\mu_d]} \\
&\qquad + (-1)^{d-1}\,\partial_\nu\omega^{a_1a_2}_{[\mu_1}\,\omega^{\dagger\,a_3\cdots a_d}_{\ \mu_2\cdots\mu_d]} - (-1)^{d-1}\,\partial_{[\mu_1}\omega^{a_1a_2}_{|\nu|}\,\omega^{\dagger\,a_3\cdots a_d}_{\ \mu_2\cdots\mu_d]} \\
&\qquad - (-1)^{d-1}\,\omega^{a_1a_2}_{|\nu|}\,\partial_{[\mu_1}\omega^{\dagger\,a_3\cdots a_d}_{\ \mu_2\cdots\mu_d]} - \partial_\nu\rho^{a_1a_2}\,\rho^{\dagger\,a_3\cdots a_d}_{\ \mu_1\cdots\mu_d}\Big) \\
&\qquad + \partial_\nu\xi^\sigma\,\xi^\dagger_{\sigma\mu_1\cdots\mu_d} + \partial_\sigma\big(\xi^\sigma\,\xi^\dagger_{\nu\mu_1\cdots\mu_d}\big)\ .
\end{aligned}
\tag{6.8}
$$

These expressions should be understood as valued in $\odot^\bullet_{\mathbb{R}}\mathscr{F}^\star_{\rm BV}$, but for brevity we do not write the symmetrized tensor products explicitly. The first two transformations dualize to the brackets involving the Euler–Lagrange derivatives with respect to the coframe field e and the spin connection ω respectively, and the actions of gauge transformations on them. The last two transformations dualize to the brackets involving the Noether identities corresponding to local pseudo-orthogonal and diffeomorphism gauge symmetries, and the action of gauge transformations on them. In general, the explicit proof of this dualization is a long and cumbersome calculation, which is also largely dependent on the spacetime dimension d. We illustrate how this works explicitly in Appendix B for the case $d = 4$.

7 Three-dimensional gravity

We will now specialise our discussion to the three-dimensional case, which has special features compared to higher dimensionalities. In particular, in this case the pertinent L_∞-algebra is a differential graded Lie algebra. Using the special feature of general relativity in three spacetime dimensions, where it defines a topological field theory in the sense that it is absent of local propagating degrees of freedom, we can make contact with the L_∞-algebra formulations of Section 4 through explicit L_∞-quasi-isomorphisms. This will show, in particular, that in the L_∞-algebra framework, three-dimensional gravity (including degenerate metrics) is pertubatively *off-shell* equivalent to a Chern–Simons gauge theory, extending the well-known on-shell equivalence [57]. This result may be interepreted as an extension of the recent analogue result which applies to the strictly non-degenerate sector [30], phrased in the dual BV-BRST framework.

7.1 Field equations

Specialising the discussion of Section 3, the Einstein–Cartan–Palatini action functional for gravity in $d = 3$ dimensions including cosmological constant is given by

$$
\begin{aligned}
S_{\rm ECP}(e,\omega) :&= \int_M \operatorname{Tr}\Big(e \curlywedge R + \frac{\Lambda}{3}\, e \curlywedge e \curlywedge e\Big) \\
&= \int_M \operatorname{Tr}\Big(\Big(e^a \wedge R^{bc} + \frac{\Lambda}{3}\, e^a \wedge e^b \wedge e^c\Big)\, \mathtt{E}_a \wedge \mathtt{E}_b \wedge \mathtt{E}_c\Big) \\
&= \int_M \varepsilon_{abc}\,\Big(e^a \wedge R^{bc} + \frac{\Lambda}{3}\, e^a \wedge e^b \wedge e^c\Big) \ ,
\end{aligned}
\tag{7.1}
$$

where ε_{abc} is the Levi–Civita tensor. In this section we shall work in Lorentzian signature $(p,q) = (1,2)$ for definiteness, but our considerations apply equally well in Euclidean signature without substantial change. The field equations are

$$
R + \Lambda\, e \curlywedge e = 0 \qquad \text{and} \qquad T = 0 \ . \tag{7.2}
$$

When e is invertible, these are equivalent to the three-dimensional Einstein field equations up to local $\mathsf{SO}_+(1,2)$ Lorentz transformations, whose solutions are spacetimes of constant curvature. This means that there are no gravitational waves on three-dimensional spacetimes, but even though all classical spacetimes obtained as solutions to (7.2) are locally gauge equivalent, they can have different topology [57].

7.2 L_∞-algebra formulation

The L_∞-algebra corresponding to the gravity theory in $d = 3$ dimensions from Section 7.1 is given by the vector space

$$
V := V_0 \oplus V_1 \oplus V_2 \oplus V_3 \tag{7.3}
$$

where

$$
\begin{aligned}
V_0 &= \Gamma(TM) \times \Omega^0\big(M, \mathfrak{so}(1,2)\big) \ , \\
V_1 &= \Omega^1(M, \mathbb{R}^{1,2}) \times \Omega^1\big(M, \mathfrak{so}(1,2)\big) \ , \\
V_2 &= \Omega^2\big(M, \textstyle\bigwedge^2(\mathbb{R}^{1,2})\big) \times \Omega^2(M, \mathbb{R}^{1,2}) \ , \\
V_3 &= \Omega^1\big(M, \Omega^3(M)\big) \times \Omega^3(M, \mathbb{R}^{1,2}) \ .
\end{aligned}
\tag{7.4}
$$

The brackets of Section 5.1 reduce in this case to

$$
\begin{aligned}
\ell_1(\xi,\rho) &= (0,\mathrm{d}\rho) \ \in\ V_1 \ , \\
\ell_1(e,\omega) &= (\mathrm{d}\omega,\mathrm{d}e) \ \in\ V_2 \ , \\
\ell_1(E,\Omega) &= (0,\mathrm{d}\Omega) \ \in\ V_3 \ , \\
\ell_2\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\big) &= \big([\xi_1,\xi_2]\,,\,-[\rho_1,\rho_2]+\xi_1(\rho_2)-\xi_2(\rho_1)\big) \ \in\ V_0 \ , \\
\ell_2\big((\xi,\rho)\,,\,(e,\omega)\big) &= \big(-\rho\cdot e+\mathrm{L}_\xi e\,,\,-[\rho,\omega]+\mathrm{L}_\xi\omega\big) \ \in\ V_1 \ , \\
\ell_2\big((\xi,\rho)\,,\,(E,\Omega)\big) &= \big(-[\rho,E]+\mathrm{L}_\xi E\,,\,-\rho\cdot\Omega+\mathrm{L}_\xi\Omega\big) \ \in\ V_2 \ , \\
\ell_2\big((\xi,\rho)\,,\,(\mathcal{X},\mathcal{P})\big) &= \big(\mathrm{d}x^\mu\otimes\mathrm{Tr}(\iota_\mu\mathrm{d}\rho\curlywedge\mathcal{P})+\mathrm{L}_\xi\mathcal{X}\,,\,-\rho\cdot\mathcal{P}+\mathrm{L}_\xi\mathcal{P}\big) \ \in\ V_3 \ , \\
\ell_2\big((e_1,\omega_1)\,,\,(e_2,\omega_2)\big) &= -\big([\omega_1,\omega_2]+2\,\Lambda\,e_2\curlywedge e_1\,,\,\omega_1\wedge e_2+\omega_2\wedge e_1\big) \ \in\ V_2 \ , \\
\ell_2\big((e,\omega)\,,\,(E,\Omega)\big) &= \Big(\mathrm{d}x^\mu\otimes\mathrm{Tr}\big(\iota_\mu\mathrm{d}e\curlywedge E+\iota_\mu\mathrm{d}\omega\curlywedge\Omega-\iota_\mu e\curlywedge\mathrm{d}E-\iota_\mu\omega\curlywedge\mathrm{d}\Omega\big)\,, \\
&\qquad\qquad E\wedge e-\omega\wedge\Omega\Big) \ \in\ V_3 \ ,
\end{aligned}
\tag{7.5}
$$

while all the rest of the brackets vanish. Thus three-dimensional gravity is organised by a differential graded Lie algebra. Again the Lie derivative L_ξ acts via the Leibniz rule on $\mathcal{X}\in\Omega^1(M)\otimes\Omega^3(M)$. The proof of the homotopy relations in this case is given in Appendix A.1.

As designed by (2.11)–(2.15), these encode the Euler–Lagrange derivatives

$$
\begin{aligned}
\mathcal{F}(e,\omega) &= (R+\Lambda\,e\curlywedge e,T) \\
&= (\mathrm{d}\omega,\mathrm{d}e)+\frac{1}{2}\,([\omega,\omega]+2\,\Lambda\,e\curlywedge e,2\,\omega\wedge e) \\
&= \ell_1(e,\omega)-\frac{1}{2}\,\ell_2\big((e,\omega)\,,\,(e,\omega)\big) \ .
\end{aligned}
$$

Moreover, the action functional (7.1) can be written as in (2.24) using the cyclic pairing (5.5):

$$
\begin{aligned}
S_{\mathrm{ECP}}(e,\omega) &= \int_M \mathrm{Tr}\Big(e\curlywedge\Big(\mathrm{d}\omega+\frac{1}{2}\,[\omega,\omega]\Big)+\frac{\Lambda}{3}\,e\curlywedge e\curlywedge e\Big) \\
&= \int_M \mathrm{Tr}\Big(\frac{1}{2}\,(e\curlywedge\mathrm{d}\omega+\omega\curlywedge\mathrm{d}e)+\frac{1}{3!}\,\big(e\curlywedge[\omega,\omega]+2\,\omega\curlywedge(\omega\wedge e)+2\,\Lambda\,e\curlywedge e\curlywedge e\big)\Big) \\
&= \frac{1}{2}\,\big\langle(e,\omega)\,,\,(\mathrm{d}\omega,\mathrm{d}e)\big\rangle+\frac{1}{3!}\,\big\langle(e,\omega)\,,\,([\omega,\omega]+2\,\Lambda\,e\curlywedge e,2\,\omega\wedge e)\big\rangle \\
&= \frac{1}{2}\,\big\langle(e,\omega)\,,\,\ell_1(e,\omega)\big\rangle-\frac{1}{3!}\,\big\langle(e,\omega)\,,\,\ell_2\big((e,\omega)\,,\,(e,\omega)\big)\big\rangle \ ,
\end{aligned}
$$

where in the second equality we integrated by parts on $e\curlywedge\mathrm{d}\omega$ and used $e\curlywedge[\omega,\omega]=\omega\curlywedge(\omega\wedge e)$ which follows from the identity (3.8).

7.3 L_∞-quasi-isomorphism with BF and Chern–Simons formulations

Three-dimensional gravity is very similar to BF theory in three dimensions: The action functional (7.1) is just the action functional (4.10) with $\mathscr{W}=\mathbb{R}^{1,2}$ and $\mathfrak{g}=\mathfrak{so}(1,2)$, along with an extra cosmological constant term. The usual shift symmetry (4.11) of BF theory correspondingly contains a slight modification to accomodate for the cosmological constant term: for $\tau \in \Omega^0(M, \mathbb{R}^{1,2})$, the action is invariant under

$$\delta_\tau e = \mathrm{d}\tau + \omega \cdot \tau \qquad \text{and} \qquad \delta_\tau \omega = 2\,\Lambda\, e \curlywedge \tau \tag{7.6}$$

with corresponding Noether identity

$$\mathrm{d}^\omega \mathcal{F}_e = \mathrm{d}^\omega R + 2\,\Lambda\, \mathrm{d}^\omega e \curlywedge e = 2\,\Lambda\, \mathcal{F}_\omega \curlywedge e$$

in $\Omega^3(M, \bigwedge^2(\mathbb{R}^{1,2}))$, where we wrote $\mathcal{F}_\omega := T = \mathrm{d}^\omega e$ and $\mathcal{F}_e := R + \Lambda\, e \curlywedge e$ as before, and used the second Bianchi identity $\mathrm{d}^\omega R = 0$. One may readily check the covariance of the Euler–Lagrange derivatives under the new transformation:

$$\delta_\tau(\mathcal{F}_e, \mathcal{F}_\omega) = (2\,\Lambda\, \mathcal{F}_\omega \curlywedge \tau\,,\, \mathcal{F}_e \wedge \tau)\ ,$$

so that in this case covariance is preserved but through a mixing of the two Euler–Lagrange derivatives.

Despite the cosmological constant modification, this symmetry may still be used to compensate for the action of any infinitesimal diffeomorphism $\xi \in \Gamma(TM)$. Indeed by choosing $\tau_\xi := \iota_\xi e$ and $\rho_\xi := \iota_\xi \omega$, one can verify

$$\delta_\xi(e, \omega) := (\mathrm{L}_\xi e, \mathrm{L}_\xi \omega) = \delta_{(\tau_\xi, \rho_\xi)}(e, \omega) + (\iota_\xi \mathcal{F}_e, \iota_\xi \mathcal{F}_\omega) \tag{7.7}$$

as in (4.14). When e is non-degenerate, that is, it is invertible as a bundle map, one may define the vector field $\xi_\tau := e^{-1}(\tau)$ that generates a shift transformation by any $\tau \in \Omega^0(M, \mathbb{R}^{1,2})$ via (7.7). Thus if one restricts the action functional (7.1) to non-degenerate coframe fields e, then the choice of generating set of gauge transformations is immaterial [57,58]. However, in the L_∞-algebra framework we need to allow for degenerate metrics in order for the space of dynamical fields to be a vector space. In this case, the two transformations are *not* equivalent, and indeed the shift symmetry generates a larger symmetry distribution on the space of fields. Hence, with this line of reasoning one should attach the extra symmetries to the cochain complex of the L_∞-algebra. Then three-dimensional gravity (including degenerate metrics) is a special case of three-dimensional BF theory, with an additional cosmological constant term.

Extending the three-dimensional ECP complex to include the extra shift symmetry and its corresponding Noether identity leads to the graded vector space

$$V_{\mathrm{ECP}}^{\mathrm{ext}} := V_0^{\mathrm{ext}} \oplus V_1^{\mathrm{ext}} \oplus V_2^{\mathrm{ext}} \oplus V_3^{\mathrm{ext}}$$

where

$$\begin{aligned} V_0^{\mathrm{ext}} &= \Gamma(TM) \times \Omega^0(M, \mathbb{R}^{1,2}) \times \Omega^0\big(M, \mathfrak{so}(1,2)\big)\ , \\ V_1^{\mathrm{ext}} = V_1 &= \Omega^1(M, \mathbb{R}^{1,2}) \times \Omega^1\big(M, \mathfrak{so}(1,2)\big)\ , \\ V_2^{\mathrm{ext}} = V_2 &= \Omega^2\big(M, \textstyle\bigwedge^2(\mathbb{R}^{1,2})\big) \times \Omega^2(M, \mathbb{R}^{1,2})\ , \\ V_3^{\mathrm{ext}} &= \Omega^1\big(M, \Omega^3(M)\big) \times \Omega^3(M, \textstyle\bigwedge^2\mathbb{R}^{1,2}) \times \Omega^3(M, \mathbb{R}^{1,2})\ , \end{aligned} \tag{7.8}$$

with the brackets extending as

$$
\begin{aligned}
\ell_1^{\rm ext}(\xi,\tau,\rho) &= (\mathrm{d}\tau,\mathrm{d}\rho) \ \in \ V_1^{\rm ext} \ , \\
\ell_1^{\rm ext}(e,\omega) &= (\mathrm{d}\omega,\mathrm{d}e) \ \in \ V_2^{\rm ext} \ , \\
\ell_1^{\rm ext}(E,\varOmega) &= (0,\mathrm{d}E,\mathrm{d}\varOmega) \ \in \ V_3^{\rm ext} \ , \\
\ell_2^{\rm ext}\big((\xi_1,\tau_1,\rho_1)\,,\,(\xi_2,\tau_2,\rho_2)\big) &= \big([\xi_1,\xi_2]\,,\,-\rho_1\cdot\tau_2+\rho_2\cdot\tau_1+\xi_1(\tau_2)-\xi_2(\tau_1)\,, \\
&\qquad\qquad -[\rho_1,\rho_2]+\xi_1(\rho_2)-\xi_2(\rho_1)\big) \ \in \ V_0^{\rm ext} \ , \\
\ell_2^{\rm ext}\big((\xi,\tau,\rho)\,,\,(e,\omega)\big) &= \big(-\rho\cdot e+\omega\cdot\tau+\mathrm{L}_\xi e\,,\,-[\rho,\omega]+2\,\Lambda\,e\curlywedge\tau+\mathrm{L}_\xi\omega\big) \ \in \ V_1^{\rm ext} \ , \\
\ell_2^{\rm ext}\big((\xi,\tau,\rho)\,,\,(E,\varOmega)\big) &= \big(-[\rho,E]+2\,\Lambda\,\varOmega\curlywedge\tau+\mathrm{L}_\xi E\,,\,-\rho\cdot\varOmega+E\wedge\tau+\mathrm{L}_\xi\varOmega\big) \ \in \ V_2^{\rm ext} \ , \\
\ell_2^{\rm ext}\big((\xi,\tau,\rho)\,,\,(\mathcal{X},\mathcal{T},\mathcal{P})\big) &= \big(\mathrm{d}x^\mu\otimes\mathrm{Tr}(\iota_\mu\mathrm{d}\rho\curlywedge\mathcal{P}+\iota_\mu\mathrm{d}\tau\curlywedge\mathcal{T})+\mathrm{L}_\xi\mathcal{X}\,, \\
&\qquad\qquad -[\rho,\mathcal{T}]+\mathrm{L}_\xi\mathcal{T}\,,\,-\rho\cdot\mathcal{P}+\mathrm{L}_\xi\mathcal{P}\big) \ \in \ V_3^{\rm ext} \ , \\
\ell_2^{\rm ext}\big((e_1,\omega_1)\,,\,(e_2,\omega_2)\big) &= -\big([\omega_1,\omega_2]+2\,\Lambda\,e_2\curlywedge e_1\,,\,\omega_1\wedge e_2+\omega_2\wedge e_1\big) \ \in \ V_2^{\rm ext} \ , \\
\ell_2^{\rm ext}\big((e,\omega)\,,\,(E,\varOmega)\big) &= \big(\mathrm{d}x^\mu\otimes\mathrm{Tr}(\iota_\mu\mathrm{d}e\curlywedge E+\iota_\mu\mathrm{d}\omega\curlywedge\varOmega-\iota_\mu e\curlywedge\mathrm{d}E-\iota_\mu\omega\curlywedge\mathrm{d}\varOmega)\,, \\
&\qquad\qquad -[\omega,E]+2\,\Lambda\,\varOmega\curlywedge e\,,\,E\wedge e-\omega\wedge\varOmega\big) \ \in \ V_3^{\rm ext} \ .
\end{aligned}
\tag{7.9}
$$

The extra homotopy relations follow from identical calculations to those of Appendix A.1. At this point one should further augment the complex by introducing a copy of the redundant part of the symmetries at degree -1 and its dual at degree 4. The final extended L_∞-algebra is then quasi-isomorphic to the L_∞-algebra constructed here but excluding the diffeomorphisms, in exactly the same way as in Section 4.4. Instead, we will circumvent this step in a more elegant way.

Recalling the equivalence between BF theory in three dimensions and Chern–Simons theory from Section 4.6, it follows that for $\Lambda = 0$ the three-dimensional ECP theory is equivalent to Chern–Simons theory based on the Lie algebra $\mathbb{R}^{1,2}\rtimes\mathfrak{so}(1,2)$. Following the observation of [57], this equivalence can be extended to $\Lambda\neq 0$, and we shall show that the extended L_∞-algebra based on (7.8) and (7.9) is isomorphic to the extended Chern–Simons L_∞-algebra based on (4.7), (4.8) and (4.9) for a special choice of Lie algebra. That is, Einstein–Cartan–Palatini theory can be formulated as a Chern–Simons gauge theory of the sort discussed in Section 4.1 whose gauge group G is the isometry group of the constant curvature three-dimensional spacetime determined by the Einstein equations (7.2): the Poincaré group $\mathsf{ISO}(1,2)=\mathbb{R}^{1,2}\rtimes\mathsf{SO}(1,2)$ for vanishing cosmological constant $\Lambda=0$, the de Sitter group $\mathsf{SO}(1,3)$ for $\Lambda>0$, or the anti-de Sitter group $\mathsf{SO}(2,2)$ for $\Lambda<0$. The generators P_a and $\mathsf{J}_{ab}=-\mathsf{J}_{ba}$ of the Lie algebra $\mathfrak{g}$ of G, with $a,b=1,2,3$, have Lie brackets

$$
\begin{aligned}
[\mathsf{P}_a,\mathsf{P}_b]_\mathfrak{g} &= 2\,\Lambda\,\mathsf{J}_{ab} \ , \\
[\mathsf{J}_{ab},\mathsf{P}_c]_\mathfrak{g} &= \tfrac12\,\big(\eta_{bc}\,\mathsf{P}_a-\eta_{ac}\,\mathsf{P}_b\big) \ , \\
[\mathsf{J}_{ab},\mathsf{J}_{cd}]_\mathfrak{g} &= \tfrac12\,\big(\eta_{bc}\,\mathsf{J}_{ad}-\eta_{ac}\,\mathsf{J}_{bd}+\eta_{ad}\,\mathsf{J}_{bc}-\eta_{bd}\,\mathsf{J}_{ac}\big) \ ,
\end{aligned}
$$

and there is a natural invariant quadratic form $\mathrm{Tr}_\mathfrak{g}:\mathfrak{g}\otimes\mathfrak{g}\to\mathbb{R}$ of split signature defined by [57]

$$
\mathrm{Tr}_\mathfrak{g}(\mathsf{J}_{ab}\otimes\mathsf{P}_c)=\varepsilon_{abc} \qquad \text{and} \qquad \mathrm{Tr}_\mathfrak{g}(\mathsf{J}_{ab}\otimes\mathsf{J}_{cd})=0=\mathrm{Tr}_\mathfrak{g}(\mathsf{P}_a\otimes\mathsf{P}_b) \ .
$$

By decomposing connections $A\in\Omega^1(M,\mathfrak{g})$ as

$$
A=e^a\,\mathsf{P}_a+\omega^{ab}\,\mathsf{J}_{ab} \ \in \ \Omega^1(M,\mathfrak{g}) \ , \tag{7.10}
$$

a straightforward expansion of the Chern–Simons action functional (4.1) using these commutation relations and invariant pairing shows that it coincides with the action functional (7.1) for Einstein–Cartan–Palatini gravity in three dimensions:

$$S_{\rm CS}(A) = S_{\rm ECP}(e,\omega) \ .$$

Similarly, decomposing gauge parameters $\lambda \in \Omega^0(M,\mathfrak{g})$ as

$$\lambda = \tau^a\,\mathtt{P}_a + \rho^{ab}\,\mathtt{J}_{ab} \ \in \ \Omega^0(M,\mathfrak{g}) \ , \tag{7.11}$$

a straightforward expansion again shows that the standard gauge transformations $\delta_\lambda A$ in (4.2) are equivalent to $\delta_{(\tau,\rho)}(e,\omega)$. Completely analogous statements follow for the action of diffeomorphisms, and for the forms of the Euler–Lagrange derivatives and Noether identities.

The precise statement of the equivalence above is that the two underlying extended cyclic L_∞-algebras are (strictly) isomorphic. The isomorphism is given by

$$\psi_1^{\rm ECP} : V_{\rm ECP}^{\rm ext} \longrightarrow V_{\rm CS}^{\rm ext}$$

where

$$\begin{aligned}
\psi_1^{\rm ECP}(\xi,\tau,\rho) &= (\xi, \tau^a\,\mathtt{P}_a + \rho^{ab}\,\mathtt{J}_{ab}) \ , \\
\psi_1^{\rm ECP}(e,\omega) &= e^a\,\mathtt{P}_a + \omega^{ab}\,\mathtt{J}_{ab} \ , \\
\psi_1^{\rm ECP}(E,\mathit{\Omega}) &= \mathit{\Omega}^a\,\mathtt{P}_a + E^{ab}\,\mathtt{J}_{ab} \ , \\
\psi_1^{\rm ECP}(\mathcal{X},\mathcal{T},\mathcal{P}) &= (\mathcal{X}, \mathcal{P}^a\,\mathtt{P}_a + \mathcal{T}^{ab}\,\mathtt{J}_{ab}) \ ,
\end{aligned}$$

while the remaining maps $\psi_n^{\rm ECP} : \bigwedge^n V_{\rm ECP}^{\rm ext} \to V_{\rm CS}^{\rm ext}$ are set to 0 for all $n \geq 2$. Since both sides are differential graded Lie algebras, and no higher morphisms $\psi_n^{\rm ECP}$ arise, the only relation to check is the condition that the map $\psi_1^{\rm ECP}$ is a morphism of differential graded Lie algebras. The calculations follow by a straightforward expansion of the brackets involved. Furthermore, the map is a cyclic L_∞-morphism, which follows immediately by the definition of the pairing $\mathrm{Tr}_\mathfrak{g}$.

Following the discussion of redundant symmetries in Chern–Simons theory from Section 4, one may simply drop the redundant diffeomorphism symmetries by composing with the quasi-isomorphism on the Chern–Simons side,[26] with no effect on the moduli space of classical solutions. Then this L_∞-isomorphism shows that three-dimensional ECP theory is perturbatively off-shell equivalent to Chern–Simons theory with the appropriate gauge algebra. Note that while on the gravity side the cosmological constant Λ appears explicitly in the dynamical and kinematical brackets, on the Chern–Simons side of the equivalence Λ does not appear in the definition of the n-brackets: it is reinterpreted as part of the structure constants of the chosen Lie algebra, and in this sense it is fully absorbed into the kinematical data instead.

We contrast the description we give here with that of the strong BV equivalence exhibited by [30] between three-dimensional non-degenerate ECP gravity and non-degenerate BF theory. The symplectomorphism the authors present uses explicitly the inversion property of non-degenerate coframes, and as such it cannot be straightforwardly extended to the degenerate sector. From the L_∞-algebra point of view, it is obvious we cannot intepret their map as an L_∞-morphism since restricting to non-degenerate configurations automatically takes us out of the realm of vector spaces and L_∞-algebras. Furthermore, our quasi-isomorphism may be dualised to a map that preserves the symplectic 2-form in the corresponding (degenerate) BV-BRST complexes which however will

[26]Strictly speaking we should also include vector spaces $V_{-1}^{\rm ext}$ and $V_4^{\rm ext}$ for this composition to work, but this is easily done and we do not write them explicitly.

not be a bijection. For the equivalence to be apparent one really needs to work on the underlying L_∞-algebras level, where we can invert maps -up to homotopy- which are not necessarily bijective. Another remark is that our maps point out the well known equivalence of BF and certain Chern-Simons models, thus encoding the cosmological constant in the structure Lie algebra of Chern-Simons theory as in [57]. Indeed our interpretation of redundant symmetries and use of quasi-isomorphisms, including degenerate coframes, comes more closely to that of Witten [57].

8 Four-dimensional gravity

In this final section we briefly describe the analogous four-dimensional case, wherein the L_∞-algebra formulation is no longer given by a differential graded Lie algebra.

8.1 Field equations

The discussion of Section 3 can also be specialised to yield the Einstein–Cartan–Palatini action functional for gravity in $d = 4$ dimensions including cosmological constant, which is given by

$$
\begin{aligned}
S_{\rm ECP}(e,\omega) :&= \int_M \mathrm{Tr}\Big(\frac{1}{2}\, e \curlywedge e \curlywedge R + \frac{\Lambda}{4}\, e \curlywedge e \curlywedge e \curlywedge e\Big) \\
&= \int_M \mathrm{Tr}\Big(\Big(\frac{1}{2}\, e^a \wedge e^b \wedge R^{cd} + \frac{\Lambda}{4}\, e^a \wedge e^b \wedge e^c \wedge e^d\Big)\, \mathtt{E}_a \wedge \mathtt{E}_b \wedge \mathtt{E}_c \wedge \mathtt{E}_d\Big) \\
&= \int_M \varepsilon_{abcd}\, \Big(\frac{1}{2}\, e^a \wedge e^b \wedge R^{cd} + \frac{\Lambda}{4}\, e^a \wedge e^b \wedge e^c \wedge e^d\Big) \ ,
\end{aligned}
\tag{8.1}
$$

where again we work in Lorentzian signature $(p, q) = (1, 3)$ for definiteness and identify the curvature as $R = R^{ab}\, \mathtt{E}_a \wedge \mathtt{E}_b \in \Omega^2(M, \bigwedge^2(\mathbb{R}^{1,3}))$. The field equations are now

$$
e \curlywedge R + \Lambda\, e \curlywedge e \curlywedge e = 0 \qquad \text{and} \qquad e \curlywedge T = 0 \ .
$$

In this case, when e is invertible, the second equation is equivalent to the torsion-free condition $T = 0$, because the map

$$
e \curlywedge - : \Omega^2\big(M, \mathbb{R}^{1,3}\big) \longrightarrow \Omega^3\big(M, \textstyle\bigwedge^2(\mathbb{R}^{1,3})\big)
$$

is an isomorphism. This can be used to rewrite the action functional (8.1) in terms of the curvature of the Levi–Civita connection for the metric $g = \eta_{ab}\, e^a \otimes e^b$. Then the first equation can be reduced to the Einstein equations in four dimensions (up to gauge equivalence).

8.2 L_∞-algebra formulation

Next we write out explicitly the L_∞-algebra structure of four-dimensional gravity from Section 5.1, specialised to the case $d = 4$. It is given by the vector space

$$
V := V_0 \oplus V_1 \oplus V_2 \oplus V_3
$$

where

$$
\begin{aligned}
V_0 &= \Gamma(TM) \times \Omega^0\big(M, \mathfrak{so}(1,3)\big) \ , \\
V_1 &= \Omega^1(M, \mathbb{R}^{1,3}) \times \Omega^1\big(M, \mathfrak{so}(1,3)\big) \ , \\
V_2 &= \Omega^3\big(M, \textstyle\bigwedge^3(\mathbb{R}^{1,3})\big) \times \Omega^3\big(M, \textstyle\bigwedge^2(\mathbb{R}^{1,3})\big) \ , \\
V_3 &= \Omega^1\big(M, \Omega^4(M)\big) \times \Omega^4\big(M, \textstyle\bigwedge^2(\mathbb{R}^{1,3})\big) \ .
\end{aligned}
$$

The non-vanishing brackets may be read off as follows: The 1-bracket ℓ_1 is defined by

$$
\ell_1(\xi,\rho) = (0, \mathrm{d}\rho) \ \in \ V_1 \ , \quad \ell_1(e,\omega) = (0,0) \ \in \ V_2 \qquad \text{and} \qquad \ell_1(E,\mathit{\Omega}) = (0, -\mathrm{d}\mathit{\Omega}) \ \in \ V_3 \ .
$$

The 2-bracket ℓ_2 is defined by

$$
\begin{aligned}
\ell_2\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\big) &= \big([\xi_1,\xi_2]\,,\,-[\rho_1,\rho_2] + \xi_1(\rho_2) - \xi_2(\rho_1)\big) \ \in \ V_0 \ , \\
\ell_2\big((\xi,\rho)\,,\,(e,\omega)\big) &= (-\rho\cdot e + \mathrm{L}_\xi e, -[\rho,\omega] + \mathrm{L}_\xi\omega) \ \in \ V_1 \ , \\
\ell_2\big((\xi,\rho)\,,\,(E,\mathit{\Omega})\big) &= (-\rho\cdot E + \mathrm{L}_\xi E, -[\rho,\mathit{\Omega}] + \mathrm{L}_\xi\mathit{\Omega}) \ \in \ V_2 \ , \\
\ell_2\big((\xi,\rho)\,,\,(\mathcal{X},\mathcal{P})\big) &= \big(\mathrm{d}x^\mu \otimes \mathrm{Tr}(\iota_\mu \mathrm{d}\rho \curlywedge \mathcal{P}) + \mathrm{L}_\xi\mathcal{X}\,,\,-[\rho,\mathcal{P}] + \mathrm{L}_\xi\mathcal{P}\big) \ \in \ V_3 \ , \\
\ell_2\big((e_1,\omega_1)\,,\,(e_2,\omega_2)\big) &= -(e_1 \curlywedge \mathrm{d}\omega_2 + e_2 \curlywedge \mathrm{d}\omega_1, e_1 \curlywedge \mathrm{d}e_2 + e_2 \curlywedge \mathrm{d}e_1) \ \in \ V_2 \ , \\
\ell_2\big((e,\omega)\,,\,(E,\mathit{\Omega})\big) &= \Big(\mathrm{d}x^\mu \otimes \mathrm{Tr}\big(\iota_\mu \mathrm{d}e \curlywedge E - \iota_\mu \mathrm{d}\omega \curlywedge \mathit{\Omega} - \iota_\mu e \curlywedge \mathrm{d}E + \iota_\mu\omega \curlywedge \mathrm{d}\mathit{\Omega}\big)\,, \\
&\qquad\qquad\qquad\qquad \tfrac{3}{2}\,E \wedge e + [\omega,\mathit{\Omega}]\Big) \ \in \ V_3 \ .
\end{aligned}
$$

The 3-bracket ℓ_3 is defined by

$$
\begin{aligned}
&\ell_3\big((e_1,\omega_1)\,,\,(e_2,\omega_2)\,,\,(e_3,\omega_3)\big) \\
&\quad = -\big(e_1 \curlywedge [\omega_2,\omega_3] + e_2 \curlywedge [\omega_1,\omega_3] + e_3 \curlywedge [\omega_2,\omega_1] + 3!\,\Lambda\, e_1 \curlywedge e_2 \curlywedge e_3\,, \\
&\qquad\quad e_1 \curlywedge (\omega_2 \wedge e_3) +_{(2\leftrightarrow 3)} + e_2 \curlywedge (\omega_1 \wedge e_3) +_{(1\leftrightarrow 3)} + e_3 \curlywedge (\omega_2 \wedge e_1) +_{(2\leftrightarrow 1)}\big) \ \in \ V_2 \ .
\end{aligned}
$$

The calculations establishing the homotopy relations in this case are formally identical to those of the three-dimensional case from Appendix A.1, now including an extra coframe field e where the higher brackets occur. We do not detail these cumbersome calculations and instead illustrate how the brackets follow from the dual picture of the BV–BRST formalism in Appendix B. Using the nilpotency of the BV differential this may be seen as an alternative proof of the homotopy relations.

The Euler–Lagrange derivatives are encoded in the expected way as

$$
\begin{aligned}
\mathcal{F}(e,\omega) &= \big(e \curlywedge \mathrm{d}\omega + e \curlywedge \tfrac{1}{2}\,[\omega,\omega] + \Lambda\, e^3, e \curlywedge \mathrm{d}e + e \curlywedge (\omega \wedge e)\big) \\
&= (0,0) + (e \curlywedge \mathrm{d}\omega, e \curlywedge \mathrm{d}e) + \big(e \curlywedge \tfrac{1}{2}\,[\omega,\omega] + \Lambda\, e^3, e \curlywedge (\omega \wedge e)\big) \\
&= \ell_1(e,\omega) - \frac{1}{2}\,\ell_2\big((e,\omega)\,,\,(e,\omega)\big) - \frac{1}{3!}\,\ell_3\big((e,\omega)\,,\,(e,\omega)\,,\,(e,\omega)\big) \ .
\end{aligned}
$$

The action functional (8.1) can be written as in (2.24) using the cyclic pairing (5.5) and the identity (3.8):

$$
\begin{aligned}
S_{\rm ECP}(e,\omega) &= \int_M \, {\rm Tr}\Big(\frac12\,e^2\curlywedge\Big({\rm d}\omega+\frac12\,[\omega,\omega]\Big)+\frac\Lambda4\,e^4\Big) \\
&= \int_M \, {\rm Tr}\Big(\frac1{3!}\,(e^2\curlywedge{\rm d}\omega-2\,\omega\curlywedge e\curlywedge{\rm d}e)+\frac14\,\big(e^2\curlywedge[\omega,\omega]-2\,\omega\curlywedge e\curlywedge(\omega\wedge e)+\Lambda\,e^4\big)\Big) \\
&= \frac12\,\big\langle(e,\omega)\,,\,(0,0)\big\rangle+\frac1{3!}\,\big\langle(e,\omega)\,,\,(2\,e\curlywedge{\rm d}\omega,2\,e\curlywedge{\rm d}e)\big\rangle \\
&\qquad\qquad +\frac1{4!}\,\big\langle(e,\omega)\,,\,\big(3!\,e\curlywedge\tfrac12\,[\omega,\omega]+3!\,\Lambda\,e^3,3!\,e\curlywedge(\omega\wedge e)\big)\big\rangle \\
&= \frac12\,\big\langle(e,\omega)\,,\,\ell_1(e,\omega)\big\rangle-\frac1{3!}\,\big\langle(e,\omega)\,,\,\ell_2\big((e,\omega)\,,\,(e,\omega)\big)\big\rangle \\
&\qquad\qquad -\frac1{4!}\,\big\langle(e,\omega)\,,\,\ell_3\big((e,\omega)\,,\,(e,\omega)\,,\,(e,\omega)\big)\big\rangle \ .
\end{aligned}
$$

8.3 Differential graded Lie algebra formulations

In analogy to what we did in the case of three-dimensional gravity, it is natural at this point to ask if there is an equivalent formulation of the four-dimensional ECP theory as a differential graded Lie algebra, in the sense of a quasi-isomorphism with the L_∞-algebra of Section 8.2; this would correspond to a strictification of the L_∞-algebra, which is known to exist on abstract grounds [32]. A natural place to look is again at the L_∞-algebras underlying the BF theories from Section 4.4. In particular, one may consider a four-dimensional BF theory with an additional "cosmological constant term": the action functional is given by[27]

$$
S_{\rm BF}^\Lambda(B,A)=\int_M \, {\rm Tr}\Big(B\curlywedge F+\frac\Lambda2\,B\curlywedge B\Big) \tag{8.2}
$$

for $B\in\Omega^2\big(M,\bigwedge^2(\mathbb{R}^{1,3})\big)$ and $A\in\Omega^1\big(M,\bigwedge^2(\mathbb{R}^{1,3})\big)$; equivalently, B is a two-form and A is a connection one-form both valued in the Lie algebra $\mathfrak{so}(1,3)$.

As in three spacetime dimensions, the shift symmetry survives the cosmological constant modification: for $\tau\in\Omega^1(M,\mathbb{R}^{1,3})$ we define

$$
\delta_\tau(B,A)=({\rm d}^A\tau,\Lambda\,\tau)
$$

with the corresponding Noether identity

$$
{\rm d}^A\mathcal{F}_B-\Lambda\,\mathcal{F}_A=0 \ .
$$

Again this symmetry renders diffeomorphisms redundant as in three dimensions: the shift symmetry is again large enough to kill all local degrees of freedom, so that the gauge theory is again topological. However, now the addition of a cosmological constant $\Lambda\neq0$ breaks the higher shift symmetry discussed in Section 4.5: if $\epsilon\in\Omega^0\big(M,\bigwedge^2(\mathbb{R}^{1,3})\big)$, then

$$
\big(\delta_{\tau+{\rm d}^A\epsilon}-\delta_\tau\big)(B,A)=\big((\mathcal{F}_B-\Lambda\,B)\wedge\epsilon,\Lambda\,{\rm d}^A\epsilon\big) \ \in \ V_1^{\rm BF} \ ,
$$

and this is not proportional to the Euler–Lagrange derivatives. Of course, when $\Lambda=0$ the two shift parameters $\tau+{\rm d}^A\epsilon$ and τ induce the same transformation on the space of fields, up to a term

[27] By dimension counting, such cosmological constant deformations of BF theories are only possible in dimensions $d=3,4$.

proportional to Euler–Lagrange derivatives, so that the shift symmetry is reducible as discussed in Section 4.5.

The similarity between the four-dimensional ECP action functional (8.1) and the action functional (8.2) is evident: the former is given by restricting B to decomposable diagonal two-forms in $\Omega^2\big(M,\bigwedge^2(\mathbb{R}^{1,3})\big)$. Of course, this restriction breaks the shift symmetry invariance[28] so that there are gravitational waves in four dimensions: four-dimensional gravity contains local degrees of freedom. In the L_∞-algebra framework, one can see this inequivalence between the two theories simply by noting that there does not exist a cyclic morphism between the two L_∞-algebras: This can be immediately seen at the level of the degree-preserving component $\psi_1 : V^{\rm BF} \to V^{\rm EPC}$, where the only possible non-trivial map would be of the form

$$\psi_1\big((\xi,\rho)+(e,\omega)+(E,\mathit{\Omega})+(\mathcal{X},\mathcal{P})\big) := (0,\rho)+(0,\omega)+(0,\mathit{\Omega})+(0,\mathcal{P}) \ ,$$

but this does not commute with the 1-brackets of the two sides, nor does it preserve the cyclic pairings.[29] Despite this, the relation of the Einstein–Cartan–Palatini formulation of gravity in three and four spacetime dimensions to topological field theories has prompted studies of deformations of the four-dimensional BF action functional (8.2) wherein the constraints above are implemented dynamically, and thus taking the latter action functional as a candidate for quantization, see e.g. [33]. It would be interesting and potentially fruitful to study such deformations and their explicit relations to four-dimensional gravity in the off-shell framework of L_∞-algebras, which from a mathematical perspective would give a concrete construction of the strictification of the ECP L_∞-algebra.

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Data availability. No additional research data beyond the data included and cited in this work are needed to validate the research findings presented.

A Calculations in three dimensions

In this appendix we provide various explicit calculations for three-dimensional gravity, illustrating the main steps needed to establish the L_∞-algebra relations of Section 2.1.

[28] If one restricts to nondegenerate coframe fields, one may derive a certain generalization of the three-dimensional shift symmetry [59], which is however equivalent to diffeomorphism invariance.

[29] The same is true if one tries to build a morphism in the opposite direction.

A.1 Homotopy relations

The proof of the homotopy relations for the three-dimensional Einstein–Cartan–Palatini theory of Section 7.2 is similar in spirit to the proof of the homotopy relations for three-dimensional Chern–Simons theory from [15]. In the present setting we must deal with an extra field in the fundamental representation of the gauge group and the extra diffeomorphism gauge symmetries. We check these relations order by order in the total degree of the homotopy, remembering that $|\ell_n| = 2-n$. For brevity, in the calculations below we set the cosmological constant term to zero, $\Lambda = 0$, with the results following straightforwardly for $\Lambda \neq 0$.

Differential conditions

The homotopy relations $\mathcal{J}_1 = 0$ are the differential condition $\ell_1 \circ \ell_1 = 0$. The only fields we need to check this on is a pair of gauge parameters (ξ, ρ) in degree 0:

$$\ell_1\big(\ell_1(\xi,\rho)\big) = \ell_1(0,\mathrm{d}\rho) = (0,\mathrm{d}^2\rho) = (0,0) \ ,$$

and a pair of dynamical fields (e,ω) in degree 1:

$$\ell_1\big(\ell_1(e,\omega)\big) = \ell_1(\mathrm{d}\omega,\mathrm{d}e) = (0,\mathrm{d}^2 e) = (0,0) \ .$$

Leibniz rules

The homotopy relations $\mathcal{J}_2 = 0$ are the graded Leibniz rule for the differential ℓ_1 with respect to the 2-bracket ℓ_2. In this case, we may act non-trivially on fields whose total degrees are 0, 1 and 2.

Total degree 0 : We act on two pairs of gauge transformations (ξ_1,ρ_1) and (ξ_2,ρ_2), and we need to check

$$\ell_1\Big(\ell_2\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\big)\Big) = \ell_2\big(\ell_1(\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\big) + \ell_2\big((\xi_1,\rho_1)\,,\,\ell_1(\xi_2,\rho_2)\big) \ .$$

The left-hand side is

$$\begin{aligned}\ell_1\big([\xi_1,\xi_2]\,,\,-[\rho_1,\rho_2]+\xi_1(\rho_2)-\xi_2(\rho_1)\big) &= \Big(0,-\mathrm{d}\big([\rho_1,\rho_2]+\xi_1(\rho_2)-\xi_2(\rho_1)\big)\Big) \\ &= -\big(0,[\mathrm{d}\rho_1,\rho_2]+[\rho_1,\mathrm{d}\rho_2]+\mathrm{d}\xi_1(\rho_2)-\mathrm{d}\xi_2(\rho_1)\big) \ ,\end{aligned}$$

while the right-hand side is

$$\begin{aligned}&\ell_2\big((0,\mathrm{d}\rho_1)\,,\,(\xi_2,\rho_2)\big) + \ell_2\big((\xi_1,\rho_1)\,,\,(0,\mathrm{d}\rho_2)\big) \\ &\qquad = -\big(0,-[\rho_2,\mathrm{d}\rho_1]+\mathrm{L}_{\xi_2}\mathrm{d}\rho_1\big) + \big(0,-[\rho_1,\mathrm{d}\rho_2]+\mathrm{L}_{\xi_1}\mathrm{d}\rho_2\big) \\ &\qquad = -\big(0,[\mathrm{d}\rho_1,\rho_2]+[\rho_1,\mathrm{d}\rho_2]+\mathrm{d}\xi_1(\rho_2)-\mathrm{d}\xi_2(\rho_1)\big) \ .\end{aligned}$$

Total degree 1 : We act on a pair of gauge transformations (ξ,ρ) and one pair of dynamical fields (e,ω), and we need to check

$$\ell_1\Big(\ell_2\big((e,\omega)\,,\,(\xi,\rho)\big)\Big) = \ell_2\big(\ell_1(e,\omega)\,,\,(\xi,\rho)\big) - \ell_2\big((e,\omega)\,,\,\ell_1(\xi,\rho)\big) \ .$$

The left-hand side is

$$\begin{aligned}\ell_1(\rho\cdot e-\mathrm{L}_\xi e\,,\,[\rho,\omega]-\mathrm{L}_\xi\omega)&=\big(\mathrm{d}([\rho,\omega])-\mathrm{d}\mathrm{L}_\xi\omega\,,\,\mathrm{d}(\rho\cdot e)-\mathrm{d}\mathrm{L}_\xi e\big)\\&=([\mathrm{d}\rho,\omega]+[\rho,\mathrm{d}\omega]-\mathrm{L}_\xi\mathrm{d}\omega\,,\,\mathrm{d}\rho\wedge e+\rho\cdot\mathrm{d}e-\mathrm{L}_\xi\mathrm{d}e)\end{aligned}$$

where we used the Cartan identity

$$\mathrm{d}\circ\mathrm{L}_\xi=\mathrm{L}_\xi\circ\mathrm{d}\ ,$$

while the right-hand side is

$$\ell_2\big((\mathrm{d}\omega,\mathrm{d}e)\,,\,(\xi,\rho)\big)-\ell_2\big((e,\omega)\,,\,(0,\mathrm{d}\rho)\big)=([\rho,\mathrm{d}\omega]-\mathrm{L}_\xi\mathrm{d}\omega\,,\,\rho\cdot\mathrm{d}e-\mathrm{L}_\xi\mathrm{d}e)+([\mathrm{d}\rho,\omega]\,,\,\mathrm{d}\rho\wedge e)\ .$$

<u>Total degree 2 :</u> We may act on two pairs of dynamical fields (e_1,ω_1) and (e_2,ω_2), and we need to check

$$\ell_1\Big(\ell_2\big((e_1,\omega_1)\,,\,(e_2,\omega_2)\big)\Big)=\ell_2\big(\ell_1(e_1,\omega_1)\,,\,(e_2,\omega_2)\big)-\ell_2\big((e_1,\omega_1)\,,\,\ell_1(e_2,\omega_2)\big)\ .$$

The left-hand side is

$$\ell_1\big(-([\omega_2,\omega_1],\omega_1\wedge e_2+\omega_2\wedge e_1)\big)=(0,-\mathrm{d}\omega_1\wedge e_2+\omega_1\wedge\mathrm{d}e_2-\mathrm{d}\omega_2\wedge e_1+\omega_2\wedge\mathrm{d}e_1)\ ,$$

while the right-hand side is

$$\begin{aligned}&\ell_2\big((\mathrm{d}\omega_1,\mathrm{d}e_1)\,,\,(e_2,\omega_2)\big)-\ell_2\big((e_1,\omega_1)\,,\,(\mathrm{d}\omega_2,\mathrm{d}e_2)\big)\\&\qquad=-\big(\mathrm{d}x^\mu\otimes\mathrm{Tr}(\iota_\mu\mathrm{d}e_2\curlywedge\mathrm{d}\omega_1+\iota_\mu\mathrm{d}\omega_2\curlywedge\mathrm{d}e_1)\,,\,\mathrm{d}\omega_1\wedge e_2-\omega_2\wedge\mathrm{d}e_1\big)\\&\qquad\quad-\big(\mathrm{d}x^\mu\otimes\mathrm{Tr}(\iota_\mu\mathrm{d}e_1\curlywedge\mathrm{d}\omega_2+\iota_\mu\mathrm{d}\omega_1\curlywedge\mathrm{d}e_2)\,,\,\mathrm{d}\omega_2\wedge e_1-\omega_1\wedge\mathrm{d}e_2\big)\ .\end{aligned}$$

The second component here agrees with that of the left-hand side. The first component vanishes, because we can write the argument of the Hodge duality operator using the Leibniz rule for the contraction of a spacetime four-form in three dimensions:

$$\iota_\mu\mathrm{d}e_2\curlywedge\mathrm{d}\omega_1+\iota_\mu\mathrm{d}\omega_2\curlywedge\mathrm{d}e_1+\mathrm{d}\omega_2\curlywedge\iota_\mu\mathrm{d}e_1+\mathrm{d}e_2\curlywedge\iota_\mu\mathrm{d}\omega_1=\iota_\mu(\mathrm{d}e_2\curlywedge\mathrm{d}\omega_1+\mathrm{d}\omega_2\curlywedge\mathrm{d}e_1)=0\ .$$

We may also act on a pair of gauge transformations (ξ,ρ) and a pair of Euler–Lagrange derivatives $(E,\mathit{\Omega})$. Then we need to check

$$\ell_1\Big(\ell_2\big((\xi,\rho)\,,\,(E,\mathit{\Omega})\big)\Big)-\ell_2\big(\ell_1(\xi,\rho)\,,\,(E,\mathit{\Omega})\big)-\ell_2\big((\xi,\rho)\,,\,\ell_1(E,\mathit{\Omega})\big)=(0,0)\ .$$

Expanding out the brackets we get

$$\begin{aligned}&\ell_1(-[\rho,E]+\mathrm{L}_\xi E,-\rho\cdot\mathit{\Omega}+\mathrm{L}_\xi\mathit{\Omega})-\ell_2\big((0,\mathrm{d}\rho)\,,\,(E,\mathit{\Omega})\big)-\ell_2\big((\xi,\rho)\,,\,(0,\mathrm{d}\mathit{\Omega})\big)\\&\qquad=\big(0\,,\,-\mathrm{d}(\rho\cdot\mathit{\Omega})+\mathrm{L}_\xi\mathrm{d}\mathit{\Omega}\big)-\big(\mathrm{d}x^\mu\otimes\mathrm{Tr}(-\iota_\mu\mathrm{d}\rho\curlywedge\mathrm{d}\mathit{\Omega})\,,\,-\mathrm{d}\rho\wedge\mathit{\Omega}\big)\\&\qquad\quad-\big(\mathrm{d}x^\mu\otimes\mathrm{Tr}(\iota_\mu\mathrm{d}\rho\curlywedge\mathrm{d}\mathit{\Omega})\,,\,-\rho\cdot\mathrm{d}\mathit{\Omega}+\mathrm{L}_\xi\mathrm{d}\mathit{\Omega}\big)\\&\qquad=(0,0)\ ,\end{aligned}$$

where we used the Leibniz rule for the exterior derivative in the first bracket.

Jacobi identities

Since $\ell_3=0$ by definition, the homotopy relations $\mathcal{J}_3=0$ are the graded Jacobi identity for the 2-bracket ℓ_2. In this case, we may act non-trivially on fields whose total degrees are 0, 1, 2 and 3.

Total degree 0 : We act on three pairs of gauge parameters, and we need

$$\ell_2\Big(\ell_2\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\big)\,,\,(\xi_3,\rho_3)\Big)+\ell_2\Big(\ell_2\big((\xi_3,\rho_3)\,,\,(\xi_1,\rho_1)\big)\,,\,(\xi_2,\rho_2)\Big)$$
$$+\,\ell_2\Big(\ell_2\big((\xi_2,\rho_2)\,,\,(\xi_3,\rho_3)\big)\,,\,(\xi_1,\rho_1)\Big)=(0,0)\ .$$

Expanding the first term we get

$$\ell_2\Big(\big([\xi_1,\xi_2]\,,\,-[\rho_1,\rho_2]+\xi_1(\rho_2)-\xi_2(\rho_1)\big)\,,\,(\xi_3,\rho_3)\Big)$$
$$=\Big(\big[[\xi_1,\xi_2],\xi_3\big]\,,\,\big[[\rho_1,\rho_2],\rho_3\big]-[\xi_1(\rho_2),\rho_3]+[\xi_2(\rho_1),\rho_3]+[\xi_1,\xi_2](\rho_3)$$
$$+\,\xi_3([\rho_1,\rho_2])-\xi_3\big(\xi_1(\rho_2)\big)+\xi_3\big(\xi_2(\rho_1)\big)\Big)\ .$$

Permuting the indices $(1,2,3)$ and adding, one sees that the terms containing a composition of two Lie brackets vanish due to the Jacobi identities for the Lie algebras $\Omega^0(M,\mathfrak{so}(1,2))$ and $\Gamma(TM)$. The remaining terms vanish since they form a representation of $\Gamma(TM)$ on $\Omega^0(M,\mathfrak{so}(1,2))$ and a derivation with respect to the Lie bracket of $\Omega^0(M,\mathfrak{so}(1,2))$.

Total degree 1 : We act on two pairs of gauge parameters and one pair of dynamical fields, and we need

$$\ell_2\Big(\ell_2\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\big)\,,\,(e,\omega)\Big)=\ell_2\Big(\ell_2\big((\xi_1,\rho_1)\,,\,(e,\omega)\big)\,,\,(\xi_2,\rho_2)\Big)$$
$$-\,\ell_2\Big(\ell_2\big((\xi_2,\rho_2)\,,\,(e,\omega)\big)\,,\,(\xi_1,\rho_1)\Big)\ . \tag{A.1}$$

Expanding the left-hand side we get

$$\ell_2\Big(\big([\xi_1,\xi_2]\,,\,-[\rho_1,\rho_2]+\xi_1(\rho_2)-\xi_2(\rho_1)\big)\,,\,(e,\omega)\Big)$$
$$=\big([\rho_1,\rho_2]\cdot e-\xi_1(\rho_2)\cdot e+\xi_2(\rho_1)\cdot e+\mathrm{L}_{[\xi_1,\xi_2]}e\,,$$
$$\big[[\rho_1,\rho_2],\omega\big]-[\xi_1(\rho_2),\omega]+[\xi_2(\rho_1),\omega]+\mathrm{L}_{[\xi_1,\xi_2]}\omega\big)\ ,$$

while the first term in the right-hand side is

$$\ell_2\big((-\rho_1\cdot e+\mathrm{L}_{\xi_1}e\,,\,-\,[\rho_1,\omega]+\mathrm{L}_{\xi_1}\omega)\,,\,(\xi_2,\rho_2)\big)$$
$$=\big(-\,\rho_2\cdot(\rho_1\cdot e)+\rho_2\cdot(\mathrm{L}_{\xi_1}e)+\mathrm{L}_{\xi_2}(\rho_1\cdot e)-\mathrm{L}_{\xi_2}\mathrm{L}_{\xi_1}e\,,$$
$$-\,\big[\rho_2,[\rho_1,\omega]\big]+[\rho_2,\mathrm{L}_{\xi_1}\omega]-\mathrm{L}_{\xi_2}\mathrm{L}_{\xi_1}\omega+\mathrm{L}_{\xi_2}[\rho_1,\omega]\big)\ .$$

Interchanging the indices $(1,2)$ and subtracting in this last expression, we see that the two sides of (A.1) match for the same representation theoretic reasons as in total degree 0.

Total degree 2 : We may act on collections of one pair of gauge parameters and two pairs of dynamical fields, or of two pairs of gauge parameters and one pair of Euler–Lagrange derivatives. For the former case we need

$$\ell_2\Big(\ell_2\big((e_1,\omega_1)\,,\,(e_2,\omega_2)\big)\,,\,(\xi,\rho)\Big)=-\ell_2\Big(\ell_2\big((\xi,\rho)\,,\,(e_1,\omega_1)\big)\,,\,(e_2,\omega_2)\Big)$$
$$-\,\ell_2\Big(\ell_2\big((\xi,\rho)\,,\,(e_2,\omega_2)\big)\,,\,(e_1,\omega_1)\Big)\ .$$

Expanding the left-hand side gives

$$
\begin{aligned}
-\ell_2\big(([\omega_2,\omega_1],\omega_1\wedge e_2+\omega_2\wedge e_1)\,,\,(\xi,\rho)\big)& \\
=\big(-[\rho,[\omega_2,\omega_1]]+\mathrm{L}_\xi[\omega_2,\omega_1]\,,& \\
-\rho\cdot(\omega_1\wedge e_2)-\rho\cdot(\omega_2\wedge e_1)+\mathrm{L}_\xi(\omega_1\wedge e_2)+\mathrm{L}_\xi(\omega_2\wedge e_1)\big)\ ,&
\end{aligned}
$$

while the right-hand side expands into

$$
\begin{aligned}
&\ell_2\big((\rho\cdot e_1-\mathrm{L}_\xi e_1\,,\,[\rho,\omega_1]-\mathrm{L}_\xi\omega_1)\,,\,(e_2,\omega_2)\big)+\ell_2\big((\rho\cdot e_2-\mathrm{L}_\xi e_2\,,\,[\rho,\omega_2]-\mathrm{L}_\xi\omega_2)\,,\,(e_1,\omega_1)\big)\\
&\quad=-\big([\omega_2,[\rho,\omega_1]-\mathrm{L}_\xi\omega_1]\,,\,([\rho,\omega_1]-\mathrm{L}_\xi\omega_1)\wedge e_2+\omega_2\wedge(\rho\cdot e_1-\mathrm{L}_\xi e_1)\big)\\
&\qquad-\big([\omega_1,[\rho,\omega_2]-\mathrm{L}_\xi\omega_2]\,,\,([\rho,\omega_2]-\mathrm{L}_\xi\omega_2)\wedge e_1+\omega_1\wedge(\rho\cdot e_2-\mathrm{L}_\xi e_2)\big)\\
&\quad=\big(-[\rho,[\omega_2,\omega_1]]+\mathrm{L}_\xi[\omega_2,\omega_1]\,,\\
&\qquad-\rho\cdot(\omega_1\wedge e_2)-\rho\cdot(\omega_2\wedge e_1)+\mathrm{L}_\xi(\omega_1\wedge e_2)+\mathrm{L}_\xi(\omega_2\wedge e_1)\big)
\end{aligned}
$$

as required, where in the last equality we used the Leibniz rules for the Lie derivative L_ξ and the action of the gauge parameter ρ on the exterior products $\omega\wedge e$ and $[\omega_2,\omega_1]$. The check on collections of fields involving two pairs of gauge parameters and one pair of Euler–Lagrange derivatives is formally identical to the proof of the total degree 1 relation (A.1), since the dynamical fields and the Euler–Lagrange derivatives live in the same representations of $\mathsf{SO}_+(1,2)$ and the bracket $\ell_2\big((\xi,\rho),(e,\omega)\big)$ is formally identical to $\ell_2\big((\xi,\rho),(E,\varOmega)\big)$.

Total degree 3 : The calculations now become considerably more involved and lengthy, so we will organise the checks of the graded Jacobi identities in this case into a sequence of Lemmas.

Lemma A.2 (Contracted Schouten identity). *If $A,B\in\mathfrak{so}(1,2)\simeq\bigwedge^2(\mathbb{R}^{1,2})$ and $v\in\mathbb{R}^{1,2}$, then*

$$
\varepsilon_{abc}\,A^{ab}\,B^c{}_d\,v^d=-2\,\varepsilon_{abc}\,B^a{}_d\,A^{db}\,v^c\ . \tag{A.3}
$$

Proof. We use the three-dimensional Schouten identity[30]

$$
\varepsilon_{abc}\,\eta_{dh}+\varepsilon_{ach}\,\eta_{bd}-\varepsilon_{bch}\,\eta_{ad}-\varepsilon_{abh}\,\eta_{cd}=0\ ,
$$

where η is the Minkowski metric on $\mathbb{R}^{1,2}$. This identity holds since the left-hand side is antisymmetric in four indices, whereas the indices vary through $1,2,3$ as we are working in three dimensions, and hence it vanishes identically. Contracting it with the components of $A=A^{ab}\,\mathtt{E}_{ba}$, $B=B^a{}_b\,\mathtt{E}^b{}_a$ and $v=v^a\,\mathtt{E}_a$ yields (A.3). ☐

Lemma A.4. *If (e_1,ω_1), (e_2,ω_2) and (e_3,ω_3) are three pairs of dynamical fields in V_1, then*

$$
\begin{aligned}
\ell_2\Big(\ell_2\big((e_1,\omega_1)\,,\,(e_2,\omega_2)\big)\,,\,(e_3,\omega_3)\Big)+\ell_2\Big(\ell_2\big((e_3,\omega_3)\,,\,(e_1,\omega_1)\big)\,,\,(e_2,\omega_2)\Big)&\\
+\ell_2\Big(\ell_2\big((e_2,\omega_2)\,,\,(e_3,\omega_3)\big)\,,\,(e_1,\omega_1)\Big)=(0,0)\ .&
\end{aligned} \tag{A.5}
$$

Proof. Expanding the first term of (A.5), we get

$$
\begin{aligned}
&\ell_2\big((-[\omega_2,\omega_1],-\omega_1\wedge e_2-\omega_2\wedge e_1)\,,\,(e_3,\omega_3)\big)\\
&\quad=-\Big(\mathrm{d}x^\mu\otimes\mathrm{Tr}\big(\iota_\mu\mathrm{d}e_3\curlywedge(-[\omega_2,\omega_1])+\iota_\mu\mathrm{d}\omega_3\curlywedge(-\omega_1\wedge e_2-\omega_2\wedge e_1)\\
&\qquad-\iota_\mu e_3\curlywedge\mathrm{d}(-[\omega_2,\omega_1])-\iota_\mu\omega_3\curlywedge\mathrm{d}(-\omega_1\wedge e_2-\omega_2\wedge e_1)\big)\,,\\
&\qquad-[\omega_2,\omega_1]\wedge e_3+\omega_3\wedge\omega_1\wedge e_2+\omega_3\wedge\omega_2\wedge e_1\Big)\ .
\end{aligned}
$$

[30] This identity also holds in Euclidean signature.

The other two terms are the cyclic permutations among the indices $(1,2,3)$. Writing these out one sees that the terms in the second component cancel each other as required, by representation theoretic reasons.

Showing that the first component vanishes requires a bit more work. Writing out the other two permutations among the indices $(1,2,3)$, we collect terms involving the fields ω_1, ω_2, e_3 and evaluate the Hodge operator explicitly to obtain

$$\begin{aligned}\varepsilon_{abc}\big(2\,\iota_\mu \mathrm{d}e_3{}^a \wedge \omega_2{}^b{}_d \wedge \omega_1{}^{dc} - 2\,\iota_\mu e_3{}^a\, \mathrm{d}(\omega_2{}^b{}_d \wedge \omega_1{}^{dc}) + \iota_\mu \mathrm{d}\omega_1{}^{bc} \wedge \omega_2{}^a{}_d \wedge e_3{}^d \\ - \iota_\mu \omega_1{}^{bc}\, \mathrm{d}(\omega_2{}^a{}_d \wedge e_3{}^d) + \iota_\mu \mathrm{d}\omega_2{}^{bc} \wedge \omega_1{}^a{}_d \wedge e_3{}^d - \iota_\mu \omega_2{}^{bc}\, \mathrm{d}(\omega_1{}^a{}_d \wedge e_3{}^d)\big)\ .\end{aligned} \tag{A.6}$$

We then rewrite the fourth term of (A.6) as

$$\begin{aligned}\varepsilon_{abc}\, \iota_\mu \omega_1{}^{bc}\, \mathrm{d}(\omega_2{}^a{}_d \wedge e_3{}^d) &= 2\,\varepsilon_{bca}\, \iota_\mu \omega_1{}^{bd}\, \mathrm{d}(\omega_{2d}{}^c \wedge e_3{}^a) \\ &= 2\,\varepsilon_{abc}\,(e_3{}^a \wedge \iota_\mu \omega_1{}^{bd} \wedge \mathrm{d}\omega_{2d}{}^c - \mathrm{d}e_3{}^a \wedge \iota_\mu \omega_1{}^{bd} \wedge \omega_{2d}{}^c)\end{aligned}$$

where in the first equality we used the contracted Schouten identity (A.3) and in the last equality the Leibniz rule for the exterior derivative. The sixth term is obtained from this by simply interchanging the indices $(1,2)$. For the third term of (A.6), going through exactly the same steps as for the fourth term allows us to rewrite it as

$$\iota_\mu \mathrm{d}\omega_1{}^{bc} \wedge \omega_2{}^a{}_d \wedge e_3{}^d = 2\,\varepsilon_{abc}\, e_3{}^a \wedge \iota_\mu \mathrm{d}\omega_1{}^{bd} \wedge \omega_{2d}{}^c\ ,$$

and the fifth term is simply obtained from this by interchanging the indices $(1,2)$. Collecting all the terms, the expression (A.6) becomes

$$\begin{aligned}2\,\varepsilon_{abc}\big(\iota_\mu \mathrm{d}e_3{}^a \wedge \omega_2{}^b{}_d \wedge \omega_1{}^{dc} - \iota_\mu e_3{}^a\, \mathrm{d}(\omega_2{}^b{}_d \wedge \omega_1{}^{dc}) - e_3{}^a \wedge \iota_\mu \omega_1{}^b{}_d \wedge \mathrm{d}\omega_2{}^{dc} \\ + \mathrm{d}e_3{}^a \wedge \iota_\mu \omega_1{}^b{}_d \wedge \omega_2{}^{dc} + e_3{}^a \wedge \mathrm{d}\omega_1{}^b{}_d \wedge \iota_\mu \omega_2{}^{dc} - \mathrm{d}e_3{}^a \wedge \omega_1{}^b{}_d \wedge \iota_\mu \omega_2{}^{dc} \\ + e_3{}^a \wedge \iota_\mu \mathrm{d}\omega_1{}^b{}_d \wedge \omega_2{}^{dc} + e_3{}^a \wedge \omega_1{}^b{}_d \wedge \iota_\mu \mathrm{d}\omega_2{}^{dc}\big) \\ = 2\,\varepsilon_{abc}\, \iota_\mu \mathrm{d}(e_3{}^a \wedge \omega_2{}^b{}_d \wedge \omega_1{}^{dc}) \\ = 0\end{aligned}$$

where we successively used the Leibniz rules for the exterior derivative and the contraction, and the last quantity vanishes because it is the contraction of a four-form in three dimensions. The remaining terms are simply permutations of the indices $(1,2,3)$, and so they all vanish as well. This completes the proof of the homotopy identity (A.5). ☐

Lemma A.7. *If $(\xi,\rho) \in V_0$ is a pair of gauge parameters, $(e,\omega) \in V_1$ is a pair of dynamical fields, and $(E,\varOmega) \in V_2$ is a pair of Euler–Lagrange derivatives, then*

$$\begin{aligned}\ell_2\Big(\ell_2\big((\xi,\rho)\,,\,(e,\omega)\big)\,,\,(E,\varOmega)\Big) + \ell_2\Big(\ell_2\big((E,\varOmega)\,,\,(\xi,\rho)\big)\,,\,(e,\omega)\Big) \\ + \ell_2\Big(\ell_2\big((e,\omega)\,,\,(E,\varOmega)\big)\,,\,(\xi,\rho)\Big) = (0,0)\ .\end{aligned} \tag{A.8}$$

Proof. The first term of (A.8) expands as

$$\begin{aligned}&\ell_2\big((-\rho\cdot e + \mathrm{L}_\xi e, -[\rho,\omega] + \mathrm{L}_\xi \omega)\,,\,(E,\varOmega)\big) \\ &\qquad = \Big(\mathrm{d}x^\mu \otimes \mathrm{Tr}\big(\iota_\mu \mathrm{d}(-\rho\cdot e + \mathrm{L}_\xi e) \curlywedge E + \iota_\mu \mathrm{d}(-[\rho,\omega] + \mathrm{L}_\xi \omega) \curlywedge \varOmega \\ &\qquad\qquad - \iota_\mu(-\rho\cdot e + \mathrm{L}_\xi e) \curlywedge \mathrm{d}E - \iota_\mu(-[\rho,\omega] + \mathrm{L}_\xi \omega) \curlywedge \mathrm{d}\varOmega\big)\,, \\ &\qquad\qquad E \wedge (-\rho\cdot e + \mathrm{L}_\xi e) - (-[\rho,\omega] + \mathrm{L}_\xi \omega) \wedge \varOmega\Big)\ .\end{aligned} \tag{A.9}$$

The second term expands as

$$
\begin{aligned}
-\ell_2\big((-[\rho,E]+\mathrm{L}_\xi E,\,-\rho\cdot\Omega+\mathrm{L}_\xi\Omega)\,,\,(e,\omega)\big)& \\
=\Big(\mathrm{d}x^\mu\otimes\mathrm{Tr}\big(\iota_\mu\mathrm{d}e\curlywedge(-[\rho,E]+\mathrm{L}_\xi E)&+\iota_\mu\mathrm{d}\omega\curlywedge(-\rho\cdot\Omega+\mathrm{L}_\xi\Omega) \qquad \text{(A.10)}\\
-\iota_\mu e\curlywedge\mathrm{d}(-[\rho,E]+\mathrm{L}_\xi E)&-\iota_\mu\omega\curlywedge\mathrm{d}(-\rho\cdot\Omega+\mathrm{L}_\xi\Omega)\big)\,,\\
(-[\rho,E]+\mathrm{L}_\xi E)\wedge e&-\omega\wedge(-\rho\cdot\Omega+\mathrm{L}_\xi\Omega)\Big)\ .
\end{aligned}
$$

The third term expands as

$$
\begin{aligned}
&\ell_2\Big(\big(\mathrm{d}x^\mu\otimes\mathrm{Tr}(\iota_\mu\mathrm{d}e\curlywedge E+\iota_\mu\mathrm{d}\omega\curlywedge\Omega-\iota_\mu e\curlywedge\mathrm{d}E-\iota_\mu\omega\curlywedge\mathrm{d}\Omega)\,,\,E\wedge e-\omega\wedge\Omega\big)\,,\,(\xi,\rho)\Big) \qquad \text{(A.11)}\\
&\quad=-\Big(\mathrm{d}x^\mu\otimes\mathrm{Tr}\big(\iota_\mu\mathrm{d}\rho\curlywedge(E\wedge e-\omega\wedge\Omega)\big)\\
&\qquad\qquad+\mathrm{L}_\xi\big(\mathrm{d}x^\mu\otimes(\iota_\mu\mathrm{d}e\curlywedge E-\iota_\mu e\curlywedge\mathrm{d}E+\iota_\mu\mathrm{d}\omega\curlywedge\Omega-\iota_\mu\omega\curlywedge\mathrm{d}\Omega)\big)\,,\\
&\qquad\qquad\qquad-\rho\cdot(E\wedge e-\omega\wedge\Omega)+\mathrm{L}_\xi(E\wedge e-\omega\wedge\Omega)\Big)\ .
\end{aligned}
$$

The second components in all three expanded expressions cancel each other out, as a consequence of the Leibniz rules for both gauge transformations in (ξ,ρ). Again, for the first components we need to work a bit harder.

Firstly, we collect terms in the first components involving the local Lorentz fields (ρ,e) and $\mathrm{d}E$. These amount to

$$
\mathrm{d}x^\mu\otimes\mathrm{Tr}(\rho\cdot\iota_\mu e\curlywedge\mathrm{d}E+\iota_\mu e\curlywedge\rho\cdot\mathrm{d}E)=\mathrm{d}x^\mu\otimes\mathrm{Tr}\big(\rho\cdot(\iota_\mu e\curlywedge\mathrm{d}E)\big)=0
$$

where the vanishing of the last term follows since the local infinitesimal Lorentz transformation ρ acts on the top exterior vector in $\mathbb{R}^{1,2}$, hence it is invariant under finite $\mathsf{SO}_+(1,2)$ Lorentz transformations and so the infinitesimal transformation is zero; this is exactly the same argument which shows that the action functional (7.1) is invariant under local Lorentz transformations. Similar arguments show that the terms involving (ρ,ω) and $\mathrm{d}\Omega$, $(\rho,\mathrm{d}e)$ and E, and $(\rho,\mathrm{d}\omega)$ and Ω cancel each other out.

Secondly, we collect terms involving $(\mathrm{d}\rho,e)$ and E. These amount to

$$
\begin{aligned}
&\mathrm{d}x^\mu\otimes\mathrm{Tr}\big(-\iota_\mu(\mathrm{d}\rho\wedge e)\curlywedge E+\iota_\mu e\curlywedge(\mathrm{d}\rho\wedge E)-\iota_\mu\mathrm{d}\rho\curlywedge(E\wedge e)\big)\\
&\qquad=\mathrm{d}x^\mu\otimes\mathrm{Tr}\big(\mathrm{d}\rho\wedge(\iota_\mu e\curlywedge E)\big)-\mathrm{d}x^\mu\otimes\mathrm{Tr}\big((\iota_\mu\mathrm{d}\rho)\wedge e\curlywedge E-\iota_\mu\mathrm{d}\rho\curlywedge(E\wedge e)\big)\ .
\end{aligned}
$$

The first term is zero using the $\mathsf{SO}_+(1,2)$-invariance as before. For the last two terms, we evaluate the Hodge operator explicitly to write them as

$$
\varepsilon_{abc}\,(\partial_\mu\rho^a{}_d\,e^d\wedge E^{bc}-\partial_\mu\rho^{ab}\,E^c{}_d\wedge e^d)\ .
$$

Now using the contracted Schouten identity (A.3), the first term here becomes

$$
\varepsilon_{abc}\,\partial_\mu\rho^a{}_d\,e^d\wedge E^{bc}=-2\,\varepsilon_{abc}\,\partial_\mu\rho^{ad}\,E_d{}^b\wedge e^c
$$

and similarly for the second term:

$$
-\varepsilon_{abc}\,\partial_\mu\rho^{ab}\,E^c{}_d\wedge e^d=-2\,\varepsilon_{abc}\,\partial_\mu\rho^a{}_d\,E^{db}\wedge e^c\ .
$$

Hence the two terms cancel when added. Similarly the terms involving $(\mathrm{d}\rho,\omega)$ and Ω cancel.

Thirdly, we check the terms in the first components involving (ξ, e) and E. For this, we assume that the coframe field $e = e^a\,\mathtt{E}_a$ is invertible, so that $\{e^a\}$ forms a basis for one-forms $\Omega^1(M)$.[31] The coframe basis obeys Maurer–Cartan equations

$$\mathrm{d}e^c = \tfrac{1}{2}\,k^c{}_{ab}\,e^a \wedge e^b$$

where the local structure functions $k^c{}_{ab}$ are antisymmetric in their lower indices. We can similarly write $\mathrm{d}x^\mu \otimes \iota_\mu = e^a \otimes \iota_a$, where ι_a are the contractions with vectors in the corresponding dual basis for vector fields $\Gamma(TM)$, that is, $\iota_a(e^b) = \delta^b_a$. Collecting the relevant terms from the first expansion (A.9) and expressing the arguments of the Hodge operator Tr in this basis gives

$$-\mathrm{L}_\xi e^a \otimes (\mathtt{E}_a \curlywedge \mathrm{d}E) + (\mathrm{L}_\xi k^c{}_{ab})\,e^a \otimes (e^b\,\mathtt{E}_c \curlywedge E) + k^c{}_{ab}\,\mathrm{L}_\xi e^a \otimes (e^b\,\mathtt{E}_c \curlywedge E) - k^c{}_{ba}\,e^a \otimes (\mathrm{L}_\xi e^b\,\mathtt{E}_c \curlywedge E)\ .$$

Similarly, from the second expansion (A.10) we get

$$k^c{}_{ba}\,e^b \otimes (e^a\,\mathtt{E}_c \curlywedge \mathrm{L}_\xi E) - e^b \otimes (\mathtt{E}_b \curlywedge \mathrm{d}\mathrm{L}_\xi E)$$

and lastly from the third expansion (A.11) we get

$$-\mathrm{L}_\xi(k^c{}_{ba}\,e^b) \otimes (e^a\,\mathtt{E}_c \curlywedge E) - k^c{}_{ba}\,e^b \otimes \mathrm{L}_\xi(e^a\,\mathtt{E}_c \curlywedge E) + \mathrm{L}_\xi e^b \otimes (\mathtt{E}_b \curlywedge \mathrm{d}E) + e^b \otimes (\mathtt{E}_b \curlywedge \mathrm{L}_\xi \mathrm{d}E)\ .$$

Using the Leibniz rule for the Lie derivative L_ξ and the fact that it commutes with the exterior derivative d, the third term completely cancels with the first two terms. One similarly checks the vanishing of the terms containing (ξ, ω) and Ω. This completes the proof of the homotopy identity (A.8). ☐

Lemma A.12. *If (ξ_1, ρ_1) and (ξ_2, ρ_2) are two pairs of gauge transformations in V_0, and $(\mathcal{X}, \mathcal{P}) \in V_3$ is a pair of Noether identities, then*

$$\ell_2\Big(\ell_2\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\big)\,,\,(\mathcal{X},\mathcal{P})\Big) + \ell_2\Big(\ell_2\big((\mathcal{X},\mathcal{P})\,,\,(\xi_1,\rho_1)\big)\,,\,(\xi_2,\rho_2)\Big) \\ + \ell_2\Big(\ell_2\big((\xi_2,\rho_2)\,,\,(\mathcal{X},\mathcal{P})\big)\,,\,(\xi_1,\rho_1)\Big) = (0,0)\ . \qquad \text{(A.13)}$$

Proof. The first term of (A.13) expands as

$$\ell_2\big(([\xi_1,\xi_2], -[\rho_1,\rho_2] + \mathrm{L}_{\xi_1}\rho_2 - \mathrm{L}_{\xi_2}\rho_1)\,,\,(\mathcal{X},\mathcal{P})\big) \\ = \Big(\mathrm{d}x^\mu \otimes \mathrm{Tr}\big(\partial_\mu(-[\rho_1,\rho_2] + \mathrm{L}_{\xi_1}\rho_2 - \mathrm{L}_{\xi_2}\rho_1) \curlywedge \mathcal{P}\big) + \mathrm{L}_{[\xi_1,\xi_2]}\mathcal{X}\,, \\ -(-[\rho_1,\rho_2] + \mathrm{L}_{\xi_1}\rho_2 - \mathrm{L}_{\xi_2}\rho_1)\cdot\mathcal{P} + \mathrm{L}_{[\xi_1,\xi_2]}\mathcal{P}\Big)\ .$$

The second term expands as

$$-\ell_2\Big(\big(\mathrm{d}x^\mu \otimes \mathrm{Tr}(\partial_\mu\rho_1 \curlywedge \mathcal{P}) + \mathrm{L}_{\xi_1}\mathcal{X}, -\rho_1\cdot\mathcal{P} + \mathrm{L}_{\xi_1}\mathcal{P}\big)\,,\,(\xi_2,\rho_2)\Big) \\ = \Big(\mathrm{d}x^\mu \otimes \mathrm{Tr}\big(\partial_\mu\rho_2 \curlywedge (-\rho_1\cdot\mathcal{P} + \mathrm{L}_{\xi_1}\mathcal{P})\big) + \mathrm{L}_{\xi_2}\big(\mathrm{d}x^\mu \otimes \mathrm{Tr}(\partial_\mu\rho_1 \curlywedge \mathcal{P})\big) + \mathrm{L}_{\xi_2}\mathrm{L}_{\xi_1}\mathcal{X}\,, \\ + \rho_2\cdot(\rho_1\cdot\mathcal{P}) - \rho_2\cdot\mathrm{L}_{\xi_1}\mathcal{P} - \mathrm{L}_{\xi_2}(\rho_1\cdot\mathcal{P}) + \mathrm{L}_{\xi_2}\mathrm{L}_{\xi_1}\mathcal{P}\Big)\ .$$

The third term is just the negative of the second term with the indices $(1,2)$ interchanged. The second component of the identity (A.13) is verified by simply noting that the space $\Omega^3(M, \mathbb{R}^{1,2})$ in

[31] If $\{e^a\}$ is degenerate, one can always choose another basis and perform similar calculations. Here we make this assumption in order to streamline the computation.

which it lives is a module over the Lie algebra $\Gamma(TM) \rtimes \Omega^0\big(M, \mathfrak{so}(1,2)\big)$, just like in the proof for two pairs of gauge parameters and one pair of dynamical fields in total degree 1.

For the first component we verify this by collecting similar terms in turn, starting with the terms involving the action of two Lie derivatives:

$$\mathrm{L}_{[\xi_1,\xi_2]}\mathcal{X} + \mathrm{L}_{\xi_2}\mathrm{L}_{\xi_1}\mathcal{X} - \mathrm{L}_{\xi_1}\mathrm{L}_{\xi_2}\mathcal{X} = 0 \ ,$$

where we used the Cartan identity (5.19). Next we collect terms in the first component with one Lie derivative acting:

$$\begin{aligned}\mathrm{d}x^\mu \otimes (\partial_\mu \mathrm{L}_{\xi_1}\rho_2 \curlywedge \mathcal{P}) - \mathrm{d}x^\mu \otimes (\partial_\mu \mathrm{L}_{\xi_2}\rho_1 \curlywedge \mathcal{P}) + \mathrm{d}x^\mu \otimes (\partial_\mu \rho_2 \curlywedge \mathrm{L}_{\xi_1}\mathcal{P}) + \mathrm{L}_{\xi_2}\big(\mathrm{d}x^\mu \otimes (\partial_\mu\rho_1 \curlywedge \mathcal{P})\big) \\ - \mathrm{d}x^\mu \otimes (\partial_\mu\rho_1 \curlywedge \mathrm{L}_{\xi_2}\mathcal{P}) - \mathrm{L}_{\xi_1}\big(\mathrm{d}x^\mu \otimes (\partial_\mu\rho_2 \curlywedge \mathcal{P})\big) = 0 \ ,\end{aligned}$$

where the vanishing is easily seen by using the Leibniz rule for the Lie derivatives acting in the fourth and sixth terms, which is possible in this case as the Lie derivatives here act on functions. Lastly we collect terms with no Lie derivatives acting, which are the leftover terms involving ρ_1 and ρ_2:

$$-\mathrm{d}x^\mu \otimes \mathrm{Tr}\big(\partial_\mu([\rho_1,\rho_2]) \curlywedge \mathcal{P}\big) - \mathrm{d}x^\mu \otimes \mathrm{Tr}\big(\partial_\mu\rho_2 \curlywedge \rho_1 \cdot \mathcal{P}\big) + \mathrm{d}x^\mu \otimes \mathrm{Tr}\big(\partial_\mu\rho_1 \curlywedge \rho_2 \cdot \mathcal{P}\big) \ .$$

Applying the derivatives ∂_μ using the Leibniz rule, the terms involving $\partial_\mu\rho_2$ give

$$-\mathrm{d}x^\mu \otimes \mathrm{Tr}\big([\rho_1, \partial_\mu\rho_2] \curlywedge \mathcal{P}\big) - \mathrm{d}x^\mu \otimes \mathrm{Tr}\big(\partial_\mu\rho_2 \curlywedge \rho_1 \cdot \mathcal{P}\big) = -\mathrm{d}x^\mu \otimes \mathrm{Tr}\big(\rho_1 \cdot (\partial_\mu\rho_2 \curlywedge \mathcal{P})\big) = 0$$

since the top exterior vector component is invariant under $\mathsf{SO}_+(1,2)$-transformations, as before. Similarly the terms involving $\partial_\mu\rho_1$ vanish, which completes the proof of the final homotopy identity (A.13) in total degree 3. □

As all higher homotopy relations vanish trivially, this completes the proof that the structure defined by (7.3)–(7.5) is indeed an L_∞-algebra.

A.2 Covariant homotopy relations

We shall now provide details of some illustrative checks of the homotopy relations for the covariant L_∞-algebra of Section 5.3 (in the case $d = 3$ and $\Lambda = 0$). Since all 1-brackets are unchanged by the covariantization, the differential conditions $\mathcal{J}_1^{\rm cov} = 0$ hold just like in the non-covariant case. On the other hand, the 2-brackets are almost all modified. For example, in the first bracket of (5.15) the Lie derivatives are removed as they would otherwise spoil the closure relation for gauge transformations, while in the sixth bracket the last term from the first component is removed by the new covariant Noether identity (5.14). It is then a straightforward proof, very similar to the non-covariant case, that the remaining modifications ensure that altogether the Leibniz rules $\mathcal{J}_2^{\rm cov} = 0$ hold.

Compared to the non-covariant case of Appendix A.1, the main new features that arise are due to the fact that the covariant L_∞-algebra for three-dimensional gravity is no longer a differential graded Lie algebra: there are new non-vanishing higher brackets (5.16) and (5.17). These incorporate the covariant gauge transformations (5.13) and their closure, together with the covariant Noether identity (5.14). Correspondingly, new higher homotopy identities should be checked, so we will focus on those.

Homotopy Jacobi identities

The modification of the 2-brackets leads to the six non-trival 3-brackets (5.16), and these have to be included when proving the homotopy relations $\mathcal{J}_3^{\rm cov}=0$, which are given in general in (2.3). The calculations are very similar to the non-covariant case, bearing in mind that some of the terms that came from non-covariant 2-brackets will now appear as a part of the covariant 3-brackets. We illustrate this by explicitly demonstrating the identity

$$\mathcal{J}_3^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\,,\,(e,\omega)\big)=(0,0)\ . \tag{A.14}$$

Firstly, we write out the homotopy relation explicitly to get

$$\begin{aligned}
&\mathcal{J}_3^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\,,\,(e,\omega)\big)\\
&\qquad=\ell_1^{\rm cov}\Big(\ell_3^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\,,\,(e,\omega)\big)\Big)+\ell_3^{\rm cov}\big(\ell_1^{\rm cov}(\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\,,\,(e,\omega)\big)\\
&\qquad\quad+\ell_3^{\rm cov}\big((\xi_1,\rho_1)\,,\,\ell_1^{\rm cov}(\xi_2,\rho_2)\,,\,(e,\omega)\big)+\ell_3^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\,,\,\ell_1^{\rm cov}(e,\omega)\big)\\
&\qquad\quad+\ell_2^{\rm cov}\Big(\ell_2^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\big)\,,\,(e,\omega)\Big)+\ell_2^{\rm cov}\Big(\ell_2^{\rm cov}\big((e,\omega)\,,\,(\xi_1,\rho_1)\big)\,,\,(\xi_2,\rho_2)\Big)\\
&\qquad\quad+\ell_2^{\rm cov}\Big(\ell_2^{\rm cov}\big((\xi_2,\rho_2)\,,\,(e,\omega)\big)\,,\,(\xi_1,\rho_1)\Big)\ .
\end{aligned}$$

Then we separately consider terms coming from 3-brackets and terms coming from 2-brackets. The non-vanishing 3-bracket terms are given by

$$\begin{aligned}
&\ell_1^{\rm cov}\big(0\,,\,-\iota_{\xi_1}\iota_{\xi_2}{\rm d}\omega\big)+\ell_3^{\rm cov}\big((0,{\rm d}\rho_1)\,,\,(\xi_2,\rho_2)\,,\,(e,\omega)\big)+\ell_3^{\rm cov}\big((\xi_1,\rho_1)\,,\,(0,{\rm d}\rho_2)\,,\,(e,\omega)\big)\\
&\qquad=\big(\iota_{\xi_2}{\rm d}\rho_1\cdot e-\iota_{\xi_1}{\rm d}\rho_2\cdot e\,,\,-{\rm d}\iota_{\xi_1}\iota_{\xi_2}{\rm d}\omega+\iota_{\xi_2}[{\rm d}\rho_1,\omega]-\iota_{\xi_1}[{\rm d}\rho_2,\omega]\big)\\
&\qquad=\big({\rm L}_{\xi_2}\rho_1\cdot e-{\rm L}_{\xi_1}\rho_2\cdot e\,,\,-{\rm d}\iota_{\xi_1}\iota_{\xi_2}{\rm d}\omega+\iota_{\xi_2}[{\rm d}\rho_1,\omega]-\iota_{\xi_1}[{\rm d}\rho_2,\omega]\big)\ ,
\end{aligned}\tag{A.15}$$

where in the last line we used $\iota_\xi{\rm d}\rho={\rm L}_\xi\rho$. The 2-bracket terms are given by

$$\begin{aligned}
&\ell_2^{\rm cov}\big(([\xi_1,\xi_2]\,,\,-[\rho_1,\rho_2])\,,\,(e,\omega)\big)-\ell_2^{\rm cov}\big((-\rho_1\cdot e+{\rm L}_{\xi_1}e\,,\,-[\rho_1,\omega]+\iota_{\xi_1}{\rm d}\omega)\,,\,(\xi_2,\rho_2)\big)\\
&\qquad\qquad+\ell_2^{\rm cov}\big((-\rho_2\cdot e+{\rm L}_{\xi_2}e\,,\,-[\rho_2,\omega]+\iota_{\xi_2}{\rm d}\omega)\,,\,(\xi_1,\rho_1)\big)\\
&\qquad=\big({\rm L}_{\xi_1}\rho_2\cdot e-{\rm L}_{\xi_2}\rho_1\cdot e\,,\,\iota_{[\xi_1,\xi_2]}{\rm d}\omega-\iota_{\xi_2}[{\rm d}\rho_1,\omega]+\iota_{\xi_1}[{\rm d}\rho_2,\omega]+\iota_{\xi_2}{\rm d}\iota_{\xi_1}{\rm d}\omega-\iota_{\xi_1}{\rm d}\iota_{\xi_2}{\rm d}\omega\big)\\
&\qquad=\big({\rm L}_{\xi_1}\rho_2\cdot e-{\rm L}_{\xi_2}\rho_1\cdot e\,,\,-{\rm d}\iota_{\xi_2}\iota_{\xi_1}{\rm d}\omega-\iota_{\xi_2}[{\rm d}\rho_1,\omega]+\iota_{\xi_1}[{\rm d}\rho_2,\omega]\big)\ ,
\end{aligned}\tag{A.16}$$

where in the last line we used the Cartan identity (5.11). Adding (A.15) and (A.16) together then proves (A.14). With similar techniques, one can show the remaining homotopy relations $\mathcal{J}_3^{\rm cov}=0$.

Higher homotopy identities

The remaining new homotopy relations to consider are $\mathcal{J}_4^{\rm cov}=0$, which are generally given by (2.4). We will only prove here the homotopy relations involving non-zero 4-brackets. Recall that there are three non-zero 4-brackets given by (5.17).

Total degree 2 : The first non-trival homotopy relation is given by

$$
\begin{aligned}
&\mathcal{J}_4^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\,,\,(e_1,\omega_1)\,,\,(e_2,\omega_2)\big)\\
&=\ell_1^{\rm cov}\Big(\ell_4^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\,,\,(e_1,\omega_1)\,,\,(e_2,\omega_2)\big)\Big) \qquad\qquad \text{(A.17)}\\
&-\ell_2^{\rm cov}\Big(\ell_3^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\,,\,(e_1,\omega_1)\big)\,,\,(e_2,\omega_2)\Big)-\ell_2^{\rm cov}\Big(\ell_3^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\,,\,(e_2,\omega_2)\big)\,,\,(e_1,\omega_1)\Big)\\
&-\ell_2^{\rm cov}\Big((\xi_1,\rho_1)\,,\,\ell_3^{\rm cov}\big((\xi_2,\rho_2)\,,\,(e_1,\omega_1)\,,\,(e_2,\omega_2)\big)\Big)+\ell_2^{\rm cov}\Big((\xi_2,\rho_2)\,,\,\ell_3^{\rm cov}\big((\xi_1,\rho_1)\,,\,(e_1,\omega_1)\,,\,(e_2,\omega_2)\big)\Big)\\
&+\ell_3^{\rm cov}\Big(\ell_2^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\big)\,,\,(e_1,\omega_1)\,,\,(e_2,\omega_2)\Big)-\ell_3^{\rm cov}\Big(\ell_2^{\rm cov}\big((\xi_1,\rho_1)\,,\,(e_1,\omega_1)\big)\,,\,(\xi_2,\rho_2)\,,\,(e_2,\omega_2)\Big)\\
&-\ell_3^{\rm cov}\Big(\ell_2^{\rm cov}\big((\xi_1,\rho_1)\,,\,(e_2,\omega_2)\big)\,,\,(\xi_2,\rho_2)\,,\,(e_1,\omega_1)\Big)\\
&-\ell_3^{\rm cov}\Big((\xi_1,\rho_1)\,,\,\ell_2^{\rm cov}\big((\xi_2,\rho_2)\,,\,(e_1,\omega_1)\big)\,,\,(e_2,\omega_2)\Big)-\ell_3^{\rm cov}\Big((\xi_1,\rho_1)\,,\,\ell_2^{\rm cov}\big((\xi_2,\rho_2)\,,\,(e_2,\omega_2)\big)\,,\,(e_1,\omega_1)\Big)
\end{aligned}
$$

where we wrote only the non-vanishing brackets. We split this long equation into three types of contributions:

$$
\mathcal{J}_4^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\,,\,(e_1,\omega_1)\,,\,(e_2,\omega_2)\big)=\mathrm{I}^{(2)}(\ell_1^{\rm cov}\circ\ell_4^{\rm cov})+\mathrm{II}^{(2)}(\ell_2^{\rm cov}\circ\ell_3^{\rm cov})+\mathrm{III}^{(2)}(\ell_3^{\rm cov}\circ\ell_2^{\rm cov})\ .
$$

The first term is given by

$$
\mathrm{I}^{(2)}=(0,\mathrm{d}\iota_{\xi_1}\iota_{\xi_2}[\omega_1,\omega_2])\ .
$$

The second term has four contributions:

$$
\mathrm{II}^{(2)}=\mathrm{II}^{(2)}_1+\mathrm{II}^{(2)}_2+\mathrm{II}^{(2)}_3+\mathrm{II}^{(2)}_4
$$

where

$$
\begin{aligned}
\mathrm{II}^{(2)}_1&=(-\iota_{\xi_1}\iota_{\xi_2}\mathrm{d}\omega_1\cdot e_2,-[\iota_{\xi_1}\iota_{\xi_2}\mathrm{d}\omega_1,\omega_2])\ ,\\
\mathrm{II}^{(2)}_2&=\mathrm{II}^{(2)}_1\,(e_1\leftrightarrow e_2,\omega_1\leftrightarrow\omega_2)\ ,\\
\mathrm{II}^{(2)}_3&=\big(-\rho_1\cdot(\iota_{\xi_2}\omega_1\cdot e_2)-\rho_1\cdot(\iota_{\xi_2}\omega_2\cdot e_1)+\mathrm{L}_{\xi_1}(\iota_{\xi_2}\omega_1\cdot e_2)+\mathrm{L}_{\xi_1}(\iota_{\xi_2}\omega_2\cdot e_1)\,,\\
&\qquad -\big[\rho_1,[\iota_{\xi_2}\omega_1,\omega_2]\big]-\big[\rho_1,[\iota_{\xi_2}\omega_2,\omega_1]\big]+\iota_{\xi_1}\mathrm{d}[\iota_{\xi_2}\omega_1,\omega_2]+\iota_{\xi_1}\mathrm{d}[\iota_{\xi_2}\omega_2,\omega_1]\big)\ ,\\
\mathrm{II}^{(2)}_4&=-\,\mathrm{II}^{(2)}_3\,(\xi_1\leftrightarrow\xi_2,\rho_1\leftrightarrow\rho_2)\ .
\end{aligned}
$$

The third term has five contributions:

$$
\mathrm{III}^{(2)}=\mathrm{III}^{(2)}_1+\mathrm{III}^{(2)}_2+\mathrm{III}^{(2)}_3+\mathrm{III}^{(2)}_4+\mathrm{III}^{(2)}_5
$$

where

$$
\begin{aligned}
\mathrm{III}^{(2)}_1&=-\big((\iota_{[\xi_1,\xi_2]}\omega_1)\cdot e_2+(\iota_{[\xi_1,\xi_2]}\omega_2)\cdot e_1\,,\,[\iota_{[\xi_1,\xi_2]}\omega_1,\omega_2]+[\iota_{[\xi_1,\xi_2]}\omega_2,\omega_1]\big)\ ,\\
\mathrm{III}^{(2)}_2&=-\big(-[\rho_1,\iota_{\xi_2}\omega_1]\cdot e_2+\iota_{\xi_2}\iota_{\xi_2}\mathrm{d}\omega_1\cdot e_2-\iota_{\xi_2}\omega_2\cdot(\rho_1\cdot e_1)+\iota_{\xi_2}\omega_2\cdot\mathrm{L}_{\xi_1}e_1\,,\\
&\qquad -\big[[\rho_1,\iota_{\xi_2}\omega_1],\omega_2\big]+[\iota_{\xi_2}\iota_{\xi_2}\mathrm{d}\omega_1,\omega_2]-\big[\iota_{\xi_2}\omega_2,[\rho_1,\omega_1]\big]+[\iota_{\xi_2}\omega_2,\iota_{\xi_1}\mathrm{d}\omega_1]\big)\ ,\\
\mathrm{III}^{(2)}_3&=\mathrm{III}^{(2)}_2\,(e_1\leftrightarrow e_2,\omega_1\leftrightarrow\omega_2)\ ,\\
\mathrm{III}^{(2)}_4&=-\,\mathrm{III}^{(2)}_2\,(\xi_1\leftrightarrow\xi_2,\rho_1\leftrightarrow\rho_2)\ ,\\
\mathrm{III}^{(2)}_5&=\mathrm{III}^{(2)}_4\,(e_1\leftrightarrow e_2,\omega_1\leftrightarrow\omega_2)\ .
\end{aligned}
$$

We now collect all the terms in the first slots of the brackets. Most of the terms cancel straightforwardly and we are left with

$$(\mathrm{L}_{\xi_1}\iota_{\xi_2}\omega_1 - \iota_{\xi_1}\mathrm{L}_{\xi_2}\omega_1)\cdot e_2 + (\mathrm{L}_{\xi_1}\iota_{\xi_2}\omega_1 - \iota_{\xi_1}\mathrm{L}_{\xi_2}\omega_2)\cdot e_1\ .$$

Using the Cartan formula for the Lie derivative and noting that $\iota_{\xi_1}\iota_{\xi_2}\omega = 0$, since ω is a one-form, we see that the remaining terms also cancel. In this way we have shown that the first slot of the brackets in (A.17) is equal to zero. Collecting all the terms in the second slots of the brackets, we notice that the double commutators combine into Jacobi identities and thus vanish. Some of the single commutators cancel straightforwardly. The remaining ones also cancel, but one has to use the Cartan formula for the Lie derivative and the identity (5.11). This completes the proof of the homotopy relation $\mathcal{J}_4^{\rm cov} = 0$ for (A.17).

<u>Total degree 4 :</u> Finally, we have to check that the homotopy relation

$$\mathcal{J}_4^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\,,\,(e,\omega)\,,\,(\mathcal{X},\mathcal{P})\big) = (0,0) \tag{A.18}$$

holds. This relation has 13 non-vanishing terms given by

$$\begin{aligned}
&\mathcal{J}_4^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\,,\,(e,\omega)\,,\,(\mathcal{X},\mathcal{P})\big)\\
&= \ell_1^{\rm cov}\Big(\ell_4^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\,,\,(e,\omega)\,,\,(\mathcal{X},\mathcal{P})\big)\Big)\\
&\quad - \ell_4^{\rm cov}\big(\ell_1^{\rm cov}(\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\,,\,(e,\omega)\,,\,(\mathcal{X},\mathcal{P})\big) - \ell_4^{\rm cov}\big((\xi_1,\rho_1)\,,\,\ell_1^{\rm cov}(\xi_2,\rho_2)\,,\,(e,\omega)\,,\,(\mathcal{X},\mathcal{P})\big)\\
&\quad - \ell_2^{\rm cov}\Big(\ell_3^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\,,\,(e,\omega)\big)\,,\,(\mathcal{X},\mathcal{P})\Big) - \ell_2^{\rm cov}\Big(\ell_3^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\,,\,(\mathcal{X},\mathcal{P})\big)\,,\,(e,\omega)\Big)\\
&\quad - \ell_2^{\rm cov}\Big((\xi_1,\rho_1)\,,\,\ell_3^{\rm cov}\big((\xi_2,\rho_2)\,,\,(e,\omega)\,,\,(\mathcal{X},\mathcal{P})\big)\Big) + \ell_2^{\rm cov}\Big((\xi_2,\rho_2)\,,\,\ell_3^{\rm cov}\big((\xi_1,\rho_1)\,,\,(e,\omega)\,,\,(\mathcal{X},\mathcal{P})\big)\Big)\\
&\quad + \ell_3^{\rm cov}\Big(\ell_2^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\big)\,,\,(e,\omega)\,,\,(\mathcal{X},\mathcal{P})\Big) - \ell_3^{\rm cov}\Big(\ell_2^{\rm cov}\big((\xi_1,\rho_1)\,,\,(e,\omega)\big)\,,\,(\xi_2,\rho_2)\,,\,(\mathcal{X},\mathcal{P})\Big)\\
&\quad - \ell_3^{\rm cov}\Big(\ell_2^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\mathcal{X},\mathcal{P})\big)\,,\,(\xi_2,\rho_2)\,,\,(e,\omega)\Big)\\
&\quad - \ell_3^{\rm cov}\Big((\xi_1,\rho_1)\,,\,\ell_2^{\rm cov}\big((\xi_2,\rho_2)\,,\,(e,\omega)\big)\,,\,(\mathcal{X},\mathcal{P})\Big) - \ell_3^{\rm cov}\Big((\xi_1,\rho_1)\,,\,\ell_2^{\rm cov}\big((\xi_2,\rho_2)\,,\,(\mathcal{X},\mathcal{P})\big)\,,\,(e,\omega)\Big)\ .
\end{aligned}$$

As previously, we group terms according to the order of brackets as

$$\mathcal{J}_4^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\,,\,(e,\omega)\,,\,(\mathcal{X},\mathcal{P})\big) = \mathrm{I}^{(4)}(\ell_1^{\rm cov}\circ\ell_4^{\rm cov}) + \mathrm{II}^{(4)}(\ell_2^{\rm cov}\circ\ell_3^{\rm cov}) + \mathrm{III}^{(4)}(\ell_3^{\rm cov}\circ\ell_2^{\rm cov})\ .$$

The first term has three contributions:

$$\mathrm{I}^{(4)} = \mathrm{I}_1^{(4)} + \mathrm{I}_2^{(4)} + \mathrm{I}_3^{(4)}$$

where

$$\begin{aligned}
\mathrm{I}_1^{(4)} &= \big(0\,,\,\mathrm{d}\omega\wedge\iota_{\xi_1}\iota_{\xi_2}\mathcal{P} - \omega\wedge\mathrm{d}\iota_{\xi_1}\iota_{\xi_2}\mathcal{P}\big)\ ,\\
\mathrm{I}_2^{(4)} &= \big(\mathrm{d}x^\mu\otimes\mathrm{Tr}(\iota_\mu\iota_{\xi_2}[\mathrm{d}\rho_1,\omega]\curlywedge\mathcal{P})\,,\,0\big)\ ,\\
\mathrm{I}_3^{(4)} &= -\mathrm{I}_2^{(4)}(\xi_1\leftrightarrow\xi_2,\rho_1\leftrightarrow\rho_2)\ .
\end{aligned}$$

The second term has four contributions:

$$\mathrm{II}^{(4)} = \mathrm{II}_1^{(4)} + \mathrm{II}_2^{(4)} + \mathrm{II}_3^{(4)} + \mathrm{II}_4^{(4)}$$

where

$$
\begin{aligned}
\mathrm{II}_1^{(4)} &= \big(0\,,\,-\iota_{\xi_1}\iota_{\xi_2}\mathrm{d}\omega\cdot\mathcal{P}\big)\ ,\\
\mathrm{II}_2^{(4)} &= -\big(\mathrm{d}x^\mu\otimes\mathrm{Tr}(\iota_\mu\mathrm{d}\omega\curlywedge\mathrm{d}\iota_{\xi_1}\iota_{\xi_2}\mathcal{P})\,,\,-\omega\wedge\mathrm{d}\iota_{\xi_1}\iota_{\xi_2}\mathcal{P}\big)\ ,\\
\mathrm{II}_3^{(4)} &= \Big(\mathrm{L}_{\xi_2}\big(\mathrm{d}x^\mu\otimes\mathrm{Tr}(\iota_\mu\iota_{\xi_1}\mathrm{d}\omega\curlywedge\mathcal{P})\big)\,,\,0\Big)\ ,\\
\mathrm{II}_4^{(4)} &= -\mathrm{II}_3^{(4)}(\xi_1\leftrightarrow\xi_2,\rho_1\leftrightarrow\rho_2)\ .
\end{aligned}
$$

Finally, the third term has five contributions:

$$
\mathrm{III}^{(4)} = \mathrm{III}_1^{(4)}+\mathrm{III}_2^{(4)}+\mathrm{III}_3^{(4)}+\mathrm{III}_4^{(4)}+\mathrm{III}_5^{(4)}
$$

where

$$
\begin{aligned}
\mathrm{III}_1^{(4)} &= \big(\mathrm{d}x^\mu\otimes\mathrm{Tr}(\iota_\mu\iota_{[\xi_1,\xi_2]}\mathrm{d}\omega\curlywedge\mathcal{P})\,,\,0\big)\ ,\\
\mathrm{III}_2^{(4)} &= \Big(\mathrm{d}x^\mu\otimes\mathrm{Tr}\big(\iota_\mu\iota_{\xi_2}\mathrm{d}(-[\rho_1,\omega]+\iota_{\xi_1}\mathrm{d}\omega)\curlywedge\mathcal{P}\big)\,,\,0\Big)\ ,\\
\mathrm{III}_3^{(4)} &= -\Big(\mathrm{d}x^\mu\otimes\mathrm{Tr}\big(\iota_\mu\iota_{\xi_2}\mathrm{d}\omega\curlywedge(\rho_1\cdot\mathcal{P})\big)\,,\,0\Big)\ ,\\
\mathrm{III}_4^{(4)} &= -\mathrm{III}_2^{(4)}(\xi_1\leftrightarrow\xi_2,\rho_1\leftrightarrow\rho_2)\ ,\\
\mathrm{III}_5^{(4)} &= -\mathrm{III}_3^{(4)}(\xi_1\leftrightarrow\xi_2,\rho_1\leftrightarrow\rho_2)\ .
\end{aligned}
$$

The terms in the second slots combine into

$$
\begin{aligned}
\mathrm{d}\omega\wedge\iota_{\xi_1}\iota_{\xi_2}\mathcal{P}-\iota_{\xi_1}\iota_{\xi_2}\mathrm{d}\omega\cdot\mathcal{P} &= \iota_{\xi_1}(\mathrm{d}\omega\wedge\iota_{\xi_2}\mathcal{P})-\iota_{\xi_1}\mathrm{d}\omega\wedge\iota_{\xi_2}\mathcal{P}+\iota_{\xi_2}\iota_{\xi_1}\mathrm{d}\omega\cdot\mathcal{P}\\
&= \iota_{\xi_1}(\mathrm{d}\omega\wedge\iota_{\xi_2}\mathcal{P})+\iota_{\xi_2}(\iota_{\xi_1}\mathrm{d}\omega\wedge\mathcal{P})\\
&= 0\ ,
\end{aligned}
$$

since the remaining two terms are contractions of four-forms, which are identically equal to zero in three dimensions. For the terms in the first slots, we split them into terms with gauge parameters ρ and terms without them. The terms with gauge parameters combine into

$$
\begin{aligned}
&\mathrm{d}x^\mu\otimes\mathrm{Tr}\big(-[\rho_1,\iota_\mu\iota_{\xi_2}\mathrm{d}\omega]\curlywedge\mathcal{P}-\iota_\mu\iota_{\xi_2}\mathrm{d}\omega\curlywedge(\rho_1\cdot\mathcal{P})+[\rho_2,\iota_\mu\iota_{\xi_1}\mathrm{d}\omega]\curlywedge\mathcal{P}\iota_\mu\iota_{\xi_1}\mathrm{d}\omega\curlywedge(\rho_2\cdot\mathcal{P})\big)\\
&\qquad= \mathrm{d}x^\mu\otimes\mathrm{Tr}\big(-\rho_1\cdot(\iota_\mu\iota_{\xi_2}\mathrm{d}\omega\curlywedge\mathcal{P})+\rho_2\cdot(\iota_\mu\iota_{\xi_1}\mathrm{d}\omega\curlywedge\mathcal{P})\big)\\
&\qquad= 0\ ,
\end{aligned}
$$

where the vanishing of the last terms follows again from the invariance of a top exterior vector under local Lorentz transformations. The terms without gauge parameters are given by

$$
\begin{aligned}
&-\mathrm{d}x^\mu\otimes\mathrm{Tr}(\iota_\mu\mathrm{d}\omega\curlywedge\mathrm{d}\iota_{\xi_1}\iota_{\xi_2}\mathcal{P})+\mathrm{L}_{\xi_2}\big(\mathrm{d}x^\mu\otimes\mathrm{Tr}(\iota_\mu\iota_{\xi_1}\mathrm{d}\omega\curlywedge\mathcal{P})\big)-\mathrm{L}_{\xi_1}\big(\mathrm{d}x^\mu\otimes\mathrm{Tr}(\iota_\mu\iota_{\xi_2}\mathrm{d}\omega\curlywedge\mathcal{P})\big)\\
&\quad+\mathrm{d}x^\mu\otimes\mathrm{Tr}(\iota_\mu\iota_{[\xi_1,\xi_2]}\mathrm{d}\omega\curlywedge\mathcal{P})+\mathrm{d}x^\mu\otimes\mathrm{Tr}\big((\iota_\mu\iota_{\xi_2}\mathrm{d}\iota_{\xi_1}\mathrm{d}\omega)\curlywedge\mathcal{P}\big)-\mathrm{d}x^\mu\otimes\mathrm{Tr}\big((\iota_\mu\iota_{\xi_1}\mathrm{d}\iota_{\xi_2}\mathrm{d}\omega)\curlywedge\mathcal{P}\big)\ .
\end{aligned}
\tag{A.19}
$$

Using the Cartan identity (5.11), the Leibniz tule for the Lie derivative and noting that

$$
(\mathrm{L}_{\xi_2}\mathrm{d}x^\mu)\otimes\mathrm{Tr}\big(\iota_\mu\iota_{\xi_1}\mathrm{d}\omega\curlywedge\mathcal{P}\big) = \mathrm{d}x^\mu\otimes\mathrm{Tr}\big(\iota_\mu\mathrm{L}_{\xi_2}\iota_{\xi_1}\mathrm{d}\omega-\mathrm{L}_{\xi_2}\iota_\mu\iota_{\xi_1}\mathrm{d}\omega\curlywedge\mathcal{P}\big)\ ,
$$

the terms (A.19) combine into

$$
\begin{aligned}
-\mathrm{d}x^\mu\otimes\mathrm{Tr}\big(&\iota_\mu\mathrm{d}\omega\curlywedge\mathrm{d}\iota_{\xi_1}\iota_{\xi_2}\mathcal{P}+\iota_\mu\iota_{\xi_1}\mathrm{d}\omega\curlywedge\mathrm{d}\iota_{\xi_2}\mathcal{P}-\iota_\mu\iota_{\xi_2}\mathrm{d}\omega\curlywedge\mathrm{d}\iota_{\xi_1}\mathcal{P}\\
&+(\iota_\mu\iota_{\xi_2}\mathrm{d}\iota_{\xi_1}\mathrm{d}\omega-\iota_\mu\iota_{\xi_1}\mathrm{d}\iota_{\xi_2}\mathrm{d}\omega-\iota_\mu\mathrm{d}\iota_{\xi_1}\iota_{\xi_2}\mathrm{d}\omega)\curlywedge\mathcal{P}\big)\\
&=-\mathrm{d}x^\mu\otimes\mathrm{Tr}\big(\iota_\mu\mathrm{d}\omega\curlywedge\iota_{[\xi_2,\xi_1]}\mathcal{P}-\iota_{[\xi_2,\xi_1]}\iota_\mu\mathrm{d}\omega\curlywedge\mathcal{P}\big)\\
&=-\mathrm{d}x^\mu\otimes\mathrm{Tr}\big(-\iota_{[\xi_2,\xi_1]}(\iota_\mu\mathrm{d}\omega\curlywedge\mathcal{P})\big)\\
&=0\ ,
\end{aligned}
$$

where the final term vanishes since it is a contraction of a four-form, which is identically zero in three dimensions. This completes the proof of the homotopy relation (A.18).

A.3 L_∞-morphism relations

We will now prove that the maps $\{\psi_n^{\rm cov}\}$ defined in Section 5.4 satisfy the L_∞-morphism relations given by (2.5) (for the case $d=3$). We shall use the facts $V^{\rm cov}=V$, $\psi_1^{\rm cov}=\mathrm{id}_V$ and $\psi_n^{\rm cov}=0$ for $n\geq 3$ implicitly with no further mention below. As with the homotopy relations, we shall proceed by degree n, remembering that $|\psi_n^{\rm cov}|=1-n$. Since $\ell_n^{\rm cov}=\ell_n$ for $n=1$ and for all $n\geq 5$, the relations are immediate and trivially satisfied in these degrees. As previously, we set $\Lambda=0$ throughout to simplify the presentation.

$n=2$: Internal degree 0

To show that the map $\psi_1^{\rm cov}$ preserves the 2-brackets up to a homotopy generated by $\psi_2^{\rm cov}$, we may act non-trivially on fields whose total degrees are 0, 1, 2 and 3.

Total degree 0 : We act on two pairs of gauge transformations (ξ_1,ρ_1) and (ξ_2,ρ_2), and we need to check

$$
\begin{aligned}
-\psi_2^{\rm cov}\big(\ell_1^{\rm cov}(\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\big)+\psi_2^{\rm cov}\big(\ell_1^{\rm cov}(\xi_2,\rho_2)\,,\,(\xi_1,\rho_1)\big)+\ell_2^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\big)&\\
=\ell_1\Big(\psi_2^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\big)\Big)+\ell_2\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\big)&\ .
\end{aligned}
$$

Expanding the left-hand side we get

$$
(0,-\iota_{\xi_2}\mathrm{d}\rho_1)+(0,\iota_{\xi_1}\mathrm{d}\rho_2)+\big([\xi_1,\xi_2],-[\rho_1,\rho_2]\big)=\big([\xi_1,\xi_2],\mathrm{L}_{\xi_1}\rho_2-\mathrm{L}_{\xi_2}\rho_1-[\rho_1,\rho_2]\big)
$$

where we used $\iota_{\xi_1}\mathrm{d}\rho_2=\mathrm{L}_{\xi_1}\rho_2$ since ρ_2 is a zero-form, and similarly for ρ_1. This is easily seen to coincide with the right-hand side, since $\psi_2^{\rm cov}((\xi_1,\rho_1),(\xi_2,\rho_2))=(0,0)$.

Total degree 1 : We act on a pair of gauge transformations (ξ,ρ) and one pair of dynamical fields (e,ω), and we need to check

$$
\begin{aligned}
-\psi_2^{\rm cov}\big(\ell_1^{\rm cov}(\xi,\rho)\,,\,(e,\omega)\big)+\psi_2^{\rm cov}\big(\ell_1^{\rm cov}(e,\omega)\,,(\xi,\rho)\big)+\ell_2^{\rm cov}\big((\xi,\rho)\,,\,(e,\omega)\big)&\\
=\ell_1\Big(\psi_2^{\rm cov}\big((\xi,\rho)\,,\,(e,\omega)\big)\Big)+\ell_2\big((\xi,\rho)\,,\,(e,\omega)\big)&\ .
\end{aligned}
$$

Expanding the left-hand side we get

$$
\begin{aligned}
-\psi_2^{\rm cov}\big((0,\mathrm{d}\rho)\,,\,(e,\omega)\big)+\psi_2^{\rm cov}\big((\mathrm{d}\omega,\mathrm{d}e)\,,\,(\xi,\rho)\big)+\big(\mathrm{L}_\xi e-\rho\cdot e\,,\,\iota_\xi\mathrm{d}\omega-[\rho,\omega]\big)&\\
=\big(\mathrm{L}_\xi e-\rho\cdot e\,,\,\iota_\xi\mathrm{d}\omega-[\rho,\omega]\big)&
\end{aligned}
$$

while the right-hand side is given by

$$(0, -\mathrm{d}\iota_\xi\omega) + \big(\mathrm{L}_\xi e - \rho\cdot e\,,\,\mathrm{L}_\xi\omega - [\rho,\omega]\big) = \big(\mathrm{L}_\xi e - \rho\cdot e\,,\,\iota_\xi\mathrm{d}\omega - [\rho,\omega]\big)$$

where we used the Cartan formula (3.4).

Total degree 2 : Checking the L_∞-morphism relation on two pairs of dynamical fields (e_1,ω_1) and (e_2,ω_2) is immediate, since all values of ψ_2^{cov} vanish by definition and $\ell_2^{\mathrm{cov}}\big((e_1,\omega_1)\,,\,(e_2,\omega_2)\big) = \ell_2\big((e_1,\omega_1)\,,\,(e_2,\omega_2)\big)$. We may also act on a pair of gauge transformations (ξ,ρ) and a pair of Euler–Lagrange derivatives $(E,\varOmega)$. Then we need to check

$$\begin{aligned}-\psi_2^{\mathrm{cov}}\big(\ell_1^{\mathrm{cov}}(\xi,\rho)\,,\,(E,\varOmega)\big) + \psi_2^{\mathrm{cov}}\big(\ell_1^{\mathrm{cov}}(E,\varOmega)\,,(\xi,\rho)\big) + \ell_2^{\mathrm{cov}}\big((\xi,\rho)\,,\,(E,\varOmega)\big)&\\ = \ell_1\Big(\psi_2^{\mathrm{cov}}\big((\xi,\rho)\,,\,(E,\varOmega)\big)\Big) + \ell_2\big((\xi,\rho)\,,\,(E,\varOmega)\big)&\ .\end{aligned}$$

Expanding the left-hand side gives

$$(0,\iota_\xi\mathrm{d}\varOmega) + (\mathrm{L}_\xi E - \rho\cdot E, \mathrm{d}\iota_\xi\varOmega - \rho\cdot\varOmega) = (\mathrm{L}_\xi E - \rho\cdot E, \mathrm{L}_\xi\varOmega - \rho\cdot\varOmega)$$

where we used the Cartan formula (3.4), which easily agrees with the right-hand side.

Total degree 3 : We may act on a pair of gauge transformations (ξ,ρ) and a pair of Noether identities $(\mathcal{X},\mathcal{P})$. We need to check

$$\begin{aligned}-\psi_2^{\mathrm{cov}}\big(\ell_1^{\mathrm{cov}}(\xi,\rho)\,,\,(\mathcal{X},\mathcal{P})\big) + \psi_2^{\mathrm{cov}}\big(\ell_1^{\mathrm{cov}}(\mathcal{X},\mathcal{P})\,,(\xi,\rho)\big) + \ell_2^{\mathrm{cov}}\big((\xi,\rho)\,,\,(\mathcal{X},\mathcal{P})\big)&\\ = \ell_1\Big(\psi_2^{\mathrm{cov}}\big((\xi,\rho)\,,\,(\mathcal{X},\mathcal{P})\big)\Big) + \ell_2\big((\xi,\rho)\,,\,(\mathcal{X},\mathcal{P})\big)&\ .\end{aligned}$$

Expanding the left-hand side we get

$$\big(\mathrm{d}x^\mu\otimes\mathrm{Tr}(\iota_\mu\mathrm{d}\rho\curlywedge\mathcal{P}),0\big) + (\mathrm{L}_\xi\mathcal{X}, -\rho\cdot\mathcal{P}) = \big(\mathrm{L}_\xi\mathcal{X} + \mathrm{d}x^\mu\otimes\mathrm{Tr}(\iota_\mu\mathrm{d}\rho\curlywedge\mathcal{P})\,,\,-\rho\cdot\mathcal{P}\big)\ ,$$

while the right-hand side is

$$(0,-\mathrm{d}\iota_\xi\mathcal{P}) + \big(\mathrm{L}_\xi\mathcal{X} + \mathrm{d}x^\mu\otimes\mathrm{Tr}(\iota_\mu\mathrm{d}\rho\curlywedge\mathcal{P})\,,\,-\rho\cdot\mathcal{P} + \mathrm{L}_\xi\mathcal{P}\big) = \big(\mathrm{L}_\xi\mathcal{X} + \mathrm{d}x^\mu\otimes\mathrm{Tr}(\iota_\mu\mathrm{d}\rho\curlywedge\mathcal{P})\,,\,-\rho\cdot\mathcal{P}\big)$$

where we used the fact that $\mathcal{P}$ is a top form, so that $\mathrm{L}_\xi\mathcal{P} = \mathrm{d}\iota_\xi\mathcal{P}$.

We may also act on a pair of dynamical fields (e,ω) and a pair of Euler–Lagrange derivatives $(E,\varOmega)$. We need to check

$$\begin{aligned}-\psi_2^{\mathrm{cov}}\big(\ell_1^{\mathrm{cov}}(e,\omega)\,,\,(E,\varOmega)\big) + \psi_2^{\mathrm{cov}}\big(\ell_1^{\mathrm{cov}}(E,\varOmega)\,,(e,\omega)\big) + \ell_2^{\mathrm{cov}}\big((e,\omega)\,,\,(E,\varOmega)\big)&\\ = \ell_1\Big(\psi_2^{\mathrm{cov}}\big((e,\omega)\,,\,(E,\varOmega)\big)\Big) + \ell_2\big((e,\omega)\,,\,(E,\varOmega)\big)&\ .\end{aligned}$$

Expanding the left-hand side gives

$$\big(\mathrm{d}x^\mu\otimes\mathrm{Tr}(-\iota_\mu\mathrm{d}\omega\curlywedge\varOmega + \iota_\mu\mathrm{d}e\curlywedge E + \iota_\mu\mathrm{d}\omega\curlywedge\varOmega - \iota_\mu e\curlywedge\mathrm{d}E)\,,\,E\wedge e - \omega\wedge\varOmega\big)$$

which easily coincides with the right-hand side.

$n = 3$: Internal degree -1

The non-trivial L_∞-morphism relations in this case comprise fields whose total degrees are 1, 2, 3 and 4. We will also implicitly use the fact that the non-covariant L_∞-algebra in three dimensions is a differential graded Lie algebra, so the bracket ℓ_3 vanishes identically.

Total degree 1 : We may act on two pairs of gauge transformations (ξ_1,ρ_1) and (ξ_2,ρ_2), and a pair of dynamical fields (e,ω). We need to check

$$\begin{aligned}\psi_2^{\rm cov}\Big(\ell_2^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\big)\,,\,(e,\omega)\Big)+\psi_2^{\rm cov}\Big(\ell_2^{\rm cov}\big((e,\omega)\,,\,(\xi_1,\rho_1)\big)\,,\,(\xi_2,\rho_2)\Big)\\ +\psi_2^{\rm cov}\Big(\ell_2^{\rm cov}\big((\xi_2,\rho_2)\,,\,(e,\omega)\big)\,,\,(\xi_1,\rho_1)\Big)+\ell_3^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\,,\,(e,\omega)\big)\\ =\ell_2\Big(\psi_1^{\rm cov}(\xi_1,\rho_1)\,,\,\psi_2^{\rm cov}\big((\xi_2,\rho_2)\,,\,(e,\omega)\big)\Big)-\ell_2\Big(\psi_1^{\rm cov}(\xi_2,\rho_2)\,,\,\psi_2^{\rm cov}\big((\xi_1,\rho_1)\,,\,(e,\omega)\big)\Big)\\ -\ell_2\Big(\psi_1^{\rm cov}(e,\omega)\,,\,\psi_2^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\big)\Big)\ .\end{aligned}$$

The left-hand side expands as

$$\begin{aligned}\big(0,-\iota_{[\xi_1,\xi_2]}\omega\big)+\big(0,-\iota_{\xi_2}\iota_{\xi_1}{\rm d}\omega+[\rho_1,\iota_{\xi_2}\omega]\big)-\big(0,-\,\iota_{\xi_1}\iota_{\xi_2}{\rm d}\omega+[\rho_2,\iota_{\xi_1}\omega]\big)+\big(0,-\iota_{\xi_1}\iota_{\xi_2}{\rm d}\omega\big)\\ =\big(0,\iota_{\xi_2}{\rm d}\iota_{\xi_1}\omega-\iota_{\xi_1}{\rm d}\iota_{\xi_2}\omega+[\rho_1,\iota_{\xi_2}\omega]-[\rho_2,\iota_{\xi_1}\omega]\big)\end{aligned}$$

where we used the Cartan identity (5.11). Expanding the right-hand side yields

$$\begin{aligned}\big(0,-{\rm L}_{\xi_1}\iota_{\xi_2}\omega+[\rho_1,\iota_{\xi_2}\omega]\big)-\big(0,-\,{\rm L}_{\xi_2}\iota_{\xi_1}\omega+[\rho_2,\iota_{\xi_1}\omega]\big)\\ =\big(0,-\iota_{\xi_1}{\rm d}\iota_{\xi_2}\omega+[\rho_1,\iota_{\xi_2}\omega]+\iota_{\xi_2}{\rm d}\iota_{\xi_1}\omega-[\rho_2,\iota_{\xi_1}\omega]\big)\end{aligned}$$

where we used ${\rm L}_{\xi_1}\iota_{\xi_2}\omega=\iota_{\xi_1}{\rm d}\iota_{\xi_2}\omega$, since ω is a one-form.

Total degree 2 : We may act on two pairs of gauge transformations (ξ_1,ρ_1) and (ξ_2,ρ_2), and a pair of Euler–Lagrange derivatives $(E,\varOmega)$; in this case all pairings appearing involve $\psi_2^{\rm cov}$ and ℓ_3, which vanish individually by definition. We may also act on a pair of gauge transformations (ξ,ρ), and two pairs of dynamical fields (e_1,ω_1) and (e_2,ω_2). Then we need to check

$$\begin{aligned}\psi_2^{\rm cov}\Big(\ell_2^{\rm cov}\big((\xi,\rho)\,,\,(e_1,\omega_1)\big)\,,\,(e_2,\omega_2)\Big)-\psi_2^{\rm cov}\Big(\ell_2^{\rm cov}\big((e_2,\omega_2)\,,\,(\xi,\rho)\big)\,,\,(e_1,\omega_1)\Big)\\ +\psi_2^{\rm cov}\Big(\ell_2^{\rm cov}\big((e_1,\omega_1)\,,\,(e_2,\omega_2)\big)\,,\,(\xi,\rho)\Big)+\ell_3^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\,,\,(e,\omega)\big)\\ =\ell_2\Big(\psi_1^{\rm cov}(\xi,\rho)\,,\,\psi_2^{\rm cov}\big((e_1,\omega_1)\,,\,(e_2,\omega_2)\big)\Big)+\ell_2\Big(\psi_1^{\rm cov}(e_1,\omega_1)\,,\,\psi_2^{\rm cov}\big((\xi,\rho)\,,\,(e_2,\omega_2)\big)\Big)\\ +\ell_2\Big(\psi_1^{\rm cov}(e_2,\omega_2)\,,\,\psi_2^{\rm cov}\big((\xi,\rho)\,,\,(e_1,\omega_1)\big)\Big)\ .\end{aligned}$$

Expanding the left-hand side we get

$$-\big(\iota_\xi\omega_1\wedge e_2+\iota_\xi\omega_2\wedge e_1\,,\,[\iota_\xi\omega_1,\omega_2]+[\iota_\xi\omega_2,\omega_1]\big)\ ,$$

which is equal to the expansion of the right-hand side.

Total degree 3 : The L_∞-morphism relation involving three pairs of dynamical fields (e_1,ω_1), (e_2,ω_2) and (e_3,ω_3) is immediate, since the brackets involved in the dynamics are identical in both versions of the theory and the map $\psi_2^{\rm cov}$ is trivial on the vectors involved. We may also act on two pairs of gauge transformations (ξ_1,ρ_1) and (ξ_2,ρ_2), and a pair of Noether identities $(\mathcal{X},\mathcal{P})$. We need to check

$$\begin{aligned}\psi_2^{\rm cov}\Big(\ell_2^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\big)\,,\,(\mathcal{X},\mathcal{P})\Big)+\psi_2^{\rm cov}\Big(\ell_2^{\rm cov}\big((\mathcal{X},\mathcal{P})\,,\,(\xi_1,\rho_1)\big)\,,\,(\xi_2,\rho_2)\Big)\\ +\psi_2^{\rm cov}\Big(\ell_2^{\rm cov}\big((\xi_2,\rho_2)\,,\,(\mathcal{X},\mathcal{P})\big)\,,\,(\xi_1,\rho_1)\Big)+\ell_3^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\,,\,(\mathcal{X},\mathcal{P})\big)\\ =\ell_2\Big(\psi_1^{\rm cov}(\xi_1,\rho_1)\,,\,\psi_2^{\rm cov}\big((\xi_2,\rho_2)\,,\,(\mathcal{X},\mathcal{P})\big)\Big)-\ell_2\Big(\psi_1^{\rm cov}(\xi_2,\rho_2)\,,\,\psi_2^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\mathcal{X},\mathcal{P})\big)\Big)\\ -\ell_2\Big(\psi_1^{\rm cov}(\mathcal{X},\mathcal{P})\,,\,\psi_2^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\big)\Big)\ .\end{aligned}$$

Expanding the left-hand side, we get

$$
\begin{aligned}
&(0,-\iota_{[\xi_1,\xi_2]}\mathcal{P})+\big(0,\iota_{\xi_2}(\rho_1\cdot\mathcal{P})\big)-\big(0,\iota_{\xi_1}(\rho_2\cdot\mathcal{P})\big)-(0,\mathrm{d}\iota_{\xi_1}\iota_{\xi_2}\mathcal{P})\\
&\qquad\qquad=\big(0,-2\,\mathrm{d}\iota_{\xi_1}\iota_{\xi_2}\mathcal{P}-\iota_{\xi_1}\mathrm{d}\iota_{\xi_2}\mathcal{P}+\iota_{\xi_2}\mathrm{d}\iota_{\xi_1}\mathcal{P}+\rho_1\cdot\iota_{\xi_2}\mathcal{P}-\rho_2\cdot\iota_{\xi_1}\mathcal{P}\big)
\end{aligned}
$$

where we used the Cartan identity (5.11). The right-hand side expands as

$$
\begin{aligned}
&\big(0,-\rho_1\cdot(-\iota_{\xi_2}\mathcal{P})+\mathrm{L}_{\xi_1}(-\iota_{\xi_2}\mathcal{P})\big)-\big(0,-\rho_2(-\iota_{\xi_1}\mathcal{P})-\mathrm{L}_{\xi_2}(-\iota_{\xi_1}\mathcal{P})\big)\\
&\qquad\qquad=\big(0,\rho_1\cdot\iota_{\xi_2}\mathcal{P}-\rho_2\cdot\iota_{\xi_1}\mathcal{P}-\iota_{\xi_1}\mathrm{d}\iota_{\xi_2}\mathcal{P}+\iota_{\xi_2}\mathrm{d}\iota_{\xi_1}\mathcal{P}-2\,\mathrm{d}\iota_{\xi_1}\iota_{\xi_2}\mathcal{P}\big)
\end{aligned}
$$

where we used Cartan's magic formula.

We may further act on a pair of gauge transformations (ξ,ρ), a pair of dynamical fields (e,ω) and a pair of Euler–Lagrange derivatives $(E,\varOmega)$. We need to check

$$
\begin{aligned}
&\psi_2^{\rm cov}\Big(\ell_2^{\rm cov}\big((\xi,\rho)\,,\,(e,\omega)\big)\,,\,(E,\varOmega)\Big)+\psi_2^{\rm cov}\Big(\ell_2^{\rm cov}\big((E,\varOmega)\,,\,(\xi,\rho)\big)\,,\,(e,\omega)\Big)\\
&\qquad+\psi_2^{\rm cov}\Big(\ell_2^{\rm cov}\big((e,\omega)\,,\,(E,\varOmega)\big)\,,\,(\xi,\rho)\Big)+\ell_3^{\rm cov}\big((\xi,\rho)\,,\,(e,\omega)\,,\,(E,\varOmega)\big)\\
&\qquad\qquad=\ell_2\Big(\psi_1^{\rm cov}(\xi,\rho)\,,\,\psi_2^{\rm cov}\big((e,\omega)\,,\,(E,\varOmega)\big)\Big)+\ell_2\Big(\psi_1^{\rm cov}(e,\omega)\,,\,\psi_2^{\rm cov}\big((\xi,\rho)\,,\,(E,\varOmega)\big)\Big)\\
&\qquad\qquad\quad+\ell_2\Big(\psi_1^{\rm cov}(E,\varOmega)\,,\,\psi_2^{\rm cov}\big((\xi,\rho)\,,\,(e,\omega)\big)\Big)\ .
\end{aligned}
$$

Expanding the left-hand side, we have

$$
\big(0,\iota_\xi(E\wedge e-\omega\wedge\varOmega)\big)+\big(-\iota_\xi\omega\cdot E,-\omega\wedge\iota_\xi\varOmega-\iota_\xi(E\wedge e)\big)=\big(-\iota_\xi\omega\cdot E,-\iota_\xi\omega\cdot\varOmega\big)
$$

where we used $\iota_\xi(\omega\wedge\varOmega)=\iota_\xi\omega\cdot\varOmega-\omega\wedge\iota_\xi\varOmega$. This easily agrees with the right-hand side.

Total degree 4 : The L_∞-morphism relation involving a pair of gauge transformations (ξ,ρ), and two pairs of Euler–Lagrange derivatives $(E_1,\varOmega_1)$ and $(E_2,\varOmega_2)$, is trivial since all terms vanish individually, by definition of $\psi_2^{\rm cov}$ and $\ell_3^{\rm cov}$. We may also act on a pair of gauge transformations (ξ,ρ), a pair of dynamical fields (e,ω) and a pair of Noether identities $(\mathcal{X},\mathcal{P})$. We need to check

$$
\begin{aligned}
&\psi_2^{\rm cov}\Big(\ell_2^{\rm cov}\big((\xi,\rho)\,,\,(e,\omega)\big)\,,\,(\mathcal{X},\mathcal{P})\Big)-\psi_2^{\rm cov}\Big(\ell_2^{\rm cov}\big((\mathcal{X},\mathcal{P})\,,\,(\xi,\rho)\big)\,,\,(e,\omega)\Big)\\
&\qquad+\psi_2^{\rm cov}\Big(\ell_2^{\rm cov}\big((e,\omega)\,,\,(\mathcal{X},\mathcal{P})\big)\,,\,(\xi,\rho)\Big)+\ell_3^{\rm cov}\big((\xi,\rho)\,,\,(e,\omega)\,,\,(\mathcal{X},\mathcal{P})\big)\\
&\qquad\qquad=\ell_2\Big(\psi_1^{\rm cov}(\xi,\rho)\,,\,\psi_2^{\rm cov}\big((e,\omega)\,,\,(\mathcal{X},\mathcal{P})\big)\Big)+\ell_2\Big(\psi_1^{\rm cov}(e,\omega)\,,\,\psi_2^{\rm cov}\big((\xi,\rho)\,,\,(\mathcal{X},\mathcal{P})\big)\Big)\\
&\qquad\qquad\quad+\ell_2\Big(\psi_1^{\rm cov}(\mathcal{X},\mathcal{P})\,,\,\psi_2^{\rm cov}\big((\xi,\rho)\,,\,(e,\omega)\big)\Big)\ .
\end{aligned}
$$

Expanding the left-hand side, we have

$$
\begin{aligned}
&\Big(-\mathrm{d}x^\mu\otimes\mathrm{Tr}\big((\iota_\mu\iota_\xi\mathrm{d}\omega-[\rho,\iota_\mu\omega])\curlywedge\mathcal{P}\big),0\Big)+\Big(-\mathrm{d}x^\mu\otimes\mathrm{Tr}\big(\iota_\mu\omega\curlywedge(-\rho\cdot\mathcal{P})\big),0\Big)\\
&\qquad\qquad+\Big(\mathrm{d}x^\mu\otimes\mathrm{Tr}\big(\iota_\mu\iota_\xi\mathrm{d}\omega\curlywedge\mathcal{P}\big),0\Big)\\
&\qquad\qquad\qquad=\Big(\mathrm{d}x^\mu\otimes\mathrm{Tr}\big(\rho\cdot(\iota_\mu\omega\curlywedge\mathcal{P})\big),0\Big)\\
&\qquad\qquad\qquad=(0,0)\ ,
\end{aligned}
$$

where we used the Leibniz rule to pull out ρ as an action on a top exterior vector-valued form, which vanishes by invariance of top exterior vectors under Lorentz transformations. On the other hand, the right-hand side expands as

$$
\begin{aligned}
\Big(- \mathrm{L}_\xi \left(\mathrm{d}x^\mu \otimes \mathrm{Tr}(\iota_\mu \omega \curlywedge \mathcal{P}) \right) , 0 \Big) - \Big(\mathrm{d}x^\mu \otimes \mathrm{Tr}\big(\iota_\mu \mathrm{d}\omega \curlywedge \iota_\xi \mathcal{P} - \iota_\mu \omega \curlywedge \mathrm{d}\iota_\xi \mathcal{P}\big) , -\omega \wedge \iota_\xi \mathcal{P} \Big) \\
+ \Big(\mathrm{d}x^\mu \otimes \mathrm{Tr}\big(\iota_\mu \mathrm{d}\iota_\xi \omega \curlywedge \mathcal{P}\big) , -\iota_\xi \omega \cdot \mathcal{P} \Big) \\
= \Big(\mathrm{d}x^\mu \otimes \mathrm{Tr}\big((-\iota_\mu \iota_\xi \mathrm{d}\omega - \iota_\mu \mathrm{d}\iota_\xi \omega) \curlywedge \mathcal{P} - \iota_\mu \omega \curlywedge \mathrm{d}\iota_\xi \mathcal{P} - \iota_\mu \mathrm{d}\omega \curlywedge \iota_\xi \mathcal{P} + \iota_\mu \omega \curlywedge \mathrm{d}\iota_\xi \mathcal{P}\big) \\
+ \mathrm{d}x^\mu \otimes \mathrm{Tr}\big(\iota_\mu \mathrm{d}\iota_\xi \omega \curlywedge \mathcal{P}\big) , \iota_\xi(\omega \wedge \mathcal{P}) \Big) \\
= \Big(\mathrm{d}x^\mu \otimes \mathrm{Tr}\big(- \iota_\mu \iota_\xi \mathrm{d}\omega \curlywedge \mathcal{P} - \iota_\mu \mathrm{d}\omega \curlywedge \iota_\xi \mathcal{P}\big) , 0 \Big) \\
= \Big(\mathrm{d}x^\mu \otimes \mathrm{Tr}\big(- \iota_\mu \iota_\xi \mathrm{d}\omega \curlywedge \mathcal{P} + \iota_\mu \iota_\xi \mathrm{d}\omega \curlywedge \mathcal{P}\big) , 0 \Big) \\
= (0,0) \ ,
\end{aligned}
$$

where in the first equality we expanded the Lie derivative using Cartan's magic formula, along with the derivation property of the contraction, and in the second equality we used $\omega \wedge \mathcal{P} = 0$ as it is a four-form in three dimensions. Then using the Cartan identity

$$
\iota_{\xi_1} \circ \iota_{\xi_2} = -\iota_{\xi_2} \circ \iota_{\xi_1}
$$

we wrote $\iota_\mu \mathrm{d}\omega \curlywedge \iota_\xi \mathcal{P} = -\iota_\xi(\iota_\mu \mathrm{d}\omega \curlywedge \mathcal{P}) + \iota_\xi \iota_\mu \mathrm{d}\omega \curlywedge \mathcal{P} = -\iota_\mu \iota_\xi \mathrm{d}\omega \curlywedge \mathcal{P}$.

Lastly, we can also act on two pairs of dynamical fields (e_1, ω_1) and (e_2, ω_2), and one pair of Euler–Lagrange derivatives $(E, \varOmega)$. We need to check

$$
\begin{aligned}
\psi_2^{\mathrm{cov}}\Big(\ell_2^{\mathrm{cov}}\big((e_1, \omega_1) , (e_2, \omega_2)\big) , (E, \varOmega)\Big) + \psi_2^{\mathrm{cov}}\Big(\ell_2^{\mathrm{cov}}\big((E, \varOmega) , (e_1, \omega_1)\big) , (e_2, \omega_2)\Big) \\
- \psi_2^{\mathrm{cov}}\Big(\ell_2^{\mathrm{cov}}\big((e_2, \omega_2) , (E, \varOmega)\big) , (e_1, \omega_1)\Big) + \ell_3^{\mathrm{cov}}\big((e_1, \omega_1) , (e_2, \omega_2) , (E, \varOmega)\big) \\
= -\ell_2\Big(\psi_1^{\mathrm{cov}}(e_1, \omega_1) , \psi_2^{\mathrm{cov}}\big((e_2, \omega_2) , (E, \varOmega)\big)\Big) - \ell_2\Big(\psi_1^{\mathrm{cov}}(e_2, \omega_2) , \psi_2^{\mathrm{cov}}\big((e_1, \omega_1) , (E, \varOmega)\big)\Big) \\
+ \ell_2\Big(\psi_1^{\mathrm{cov}}(E, \varOmega) , \psi_2^{\mathrm{cov}}\big((e_1, \omega_1) , (e_2, \omega_2)\big)\Big) \ .
\end{aligned}
$$

Expanding the left-hand side, we have

$$
\begin{aligned}
\Big(\mathrm{d}x^\mu \otimes \mathrm{Tr}\big(\iota_\mu \omega_2 \curlywedge (E \wedge e_1 - \omega_1 \wedge \varOmega)\big), 0\Big) + \Big(\mathrm{d}x^\mu \otimes \mathrm{Tr}\big(\iota_\mu \omega_1 \curlywedge (E \wedge e_2 - \omega_2 \wedge \varOmega)\big), 0\Big) \\
- \Big(\mathrm{d}x^\mu \otimes \mathrm{Tr}\big(\iota_\mu \omega_1 \curlywedge (E \wedge e_2) + \iota_\mu \omega_2 \curlywedge (E \wedge e_1) - \iota_\mu \omega_1 \curlywedge (\omega_2 \wedge \varOmega) - \iota_\mu \omega_2 \curlywedge (\omega_1 \wedge \varOmega)\big), 0\Big) \\
= (0,0) \ ,
\end{aligned}
$$

while the right-hand side vanishes since all terms vanish individually by definition of ψ_2^{cov}.

$n = 4$: Internal degree -2

The L_∞-morphism relations in this case act non-trivially on fields whose total degrees are 2, 3, 4 and 5. We will also implicitly use the fact that the non-covariant brackets ℓ_3 and ℓ_4 vanish identically. One may easily check that all terms in the identity vanish individually for all possible combinations of fields of total degree 3, so the only non-trivial checks required are in the remaining three total degrees.

Total degree 2 : When acting on three pairs of gauge transformations (ξ_1,ρ_1), (ξ_2,ρ_2) and (ξ_3,ρ_3), and a pair of Euler–Lagrange derivatives (E,Ω), all terms vanish individually by definition of $\psi_2^{\rm cov}$, $\ell_3^{\rm cov}$ and $\ell_4^{\rm cov}$. We may also act on two pairs of gauge transformations (ξ_1,ρ_1) and (ξ_2,ρ_2), and two pairs of dynamical fields (e_1,ω_1) and (e_2,ω_2). We need to check

$$
\begin{aligned}
&-\psi_2^{\rm cov}\Big(\ell_3^{\rm cov}\big((\xi_1,\rho_1),(\xi_2,\rho_2),(e_1,\omega_1)\big),(e_2,\omega_2)\Big)-\psi_2^{\rm cov}\Big(\ell_3^{\rm cov}\big((\xi_1,\rho_1),(\xi_2,\rho_2),(e_2,\omega_2)\big),(e_1,\omega_1)\Big)\\
&\quad-\psi_2^{\rm cov}\Big(\ell_3^{\rm cov}\big((\xi_1,\rho_1)\,,\,(e_1,\omega_1)\,,\,(e_2,\omega_2)\big)\,,\,(\xi_2,\rho_2)\Big)\\
&\qquad+\psi_2^{\rm cov}\Big(\ell_3^{\rm cov}\big((\xi_2,\rho_2)\,,\,(e_1,\omega_1)\,,\,(e_2,\omega_2)\big)\,,\,(\xi_1,\rho_1)\Big)+\ell_4^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\,,\,(e_1,\omega_1)\,,\,(e_2,\omega_2)\big)\\
&\qquad\quad=-\ell_2\Big(\psi_2^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\big)\,,\,\psi_2^{\rm cov}\big((e_1,\omega_1)\,,\,(e_2,\omega_2)\big)\Big)\\
&\qquad\qquad-\ell_2\Big(\psi_2^{\rm cov}\big((\xi_1,\rho_1)\,,\,(e_1,\omega_1)\big)\,,\,\psi_2^{\rm cov}\big((\xi_2,\rho_2)\,,\,(e_2,\omega_2)\big)\Big)\\
&\qquad\qquad\quad+\ell_2\Big(\psi_2^{\rm cov}\big((\xi_2,\rho_2)\,,\,(e_1,\omega_1)\big)\,,\,\psi_2^{\rm cov}\big((\xi_1,\rho_1)\,,\,(e_2,\omega_2)\big)\Big)\ .
\end{aligned}
$$

Expanding the left-hand side, we get

$$
\begin{aligned}
&-\big(0,-[\iota_{\xi_1}\omega_1,\iota_{\xi_2}\omega_2]-[\iota_{\xi_1}\omega_2,\iota_{\xi_2}\omega_1]\big)+\big(0,-[\iota_{\xi_2}\omega_1,\iota_{\xi_1}\omega_2]-[\iota_{\xi_2}\omega_2,\iota_{\xi_1}\omega_1]\big)+\big(0,\iota_{\xi_1}\iota_{\xi_2}[\omega_1,\omega_2]\big)\\
&\qquad=\big(0,-[\iota_{\xi_2}\omega_1,\iota_{\xi_1}\omega_2]-[\iota_{\xi_2}\omega_2,\iota_{\xi_1}\omega_1]\big)\ ,
\end{aligned}
$$

after expanding $\iota_{\xi_1}\iota_{\xi_2}[\omega_1,\omega_2]$ using the Leibniz rule and cancelling terms by antisymmetry. This is easily seen to be the same as the simple expansion of the right-hand side.

Total degree 4 : The only non-trivial check required in this case, whereby not every term in the identity vanishes individually, is when acting on two pairs of gauge transformations (ξ_1,ρ_1) and (ξ_2,ρ_2), a pair of dynamical fields (e,ω) and a pair of Noether identities $(\mathcal{X},\mathcal{P})$. Excluding terms that vanish by definition of $\psi_2^{\rm cov}$, it remains to check

$$
\begin{aligned}
\ell_4^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\xi_2,\rho_2)\,,\,(e,\omega)\,,\,(\mathcal{X},\mathcal{P})\big)&=-\ell_2\Big(\psi_2^{\rm cov}\big((\xi_1,\rho_1)\,,\,(e,\omega)\big)\,,\,\psi_2^{\rm cov}\big((\xi_2,\rho_2)\,,\,(\mathcal{X},\mathcal{P})\big)\Big)\\
&\quad+\ell_2\Big(\psi_2^{\rm cov}\big((\xi_2,\rho_2)\,,\,(e,\omega)\big)\,,\,\psi_2^{\rm cov}\big((\xi_1,\rho_1)\,,\,(\mathcal{X},\mathcal{P})\big)\Big)\ .
\end{aligned}
$$

The left-hand side expands as

$$
\begin{aligned}
(0,\omega\wedge\iota_{\xi_1}\iota_{\xi_2}\mathcal{P})&=\big(0,-\iota_{\xi_1}(\omega\wedge\iota_{\xi_2}\mathcal{P})+\iota_{\xi_1}\omega\cdot\iota_{\xi_2}\mathcal{P}\big)\\
&=\big(0,\iota_{\xi_1}\iota_{\xi_2}(\omega\wedge\mathcal{P})-\iota_{\xi_1}(\iota_{\xi_2}\omega\cdot\mathcal{P})+\iota_{\xi_1}\omega\cdot\iota_{\xi_2}\mathcal{P}\big)\\
&=\big(0,-\iota_{\xi_2}\omega\cdot\iota_{\xi_1}\mathcal{P}+\iota_{\xi_1}\omega\cdot\iota_{\xi_2}\mathcal{P}\big)\ ,
\end{aligned}
$$

which is precisely the expansion of the right-hand side.

Total degree 5 : Again there is only one non-trivial check required, now when acting on a pair gauge transformations (ξ,ρ), two pairs of dynamical fields (e_1,ω_1) and (e_2,ω_2), and a pair of Noether identities $(\mathcal{X},\mathcal{P})$. Excluding again terms that vanish by definition of $\psi_2^{\rm cov}$, it remains to check

$$
\begin{aligned}
&-\psi_2^{\rm cov}\Big(\ell_3\big((\xi,\rho)\,,\,(e_1,\omega_1)\,,\,(e_2,\omega_2)\big)\,,\,(\mathcal{X},\mathcal{P})\Big)+\ell_4^{\rm cov}\big((\xi,\rho)\,,\,(e_1,\omega_1)\,,\,(e_2,\omega_2)\,,\,(\mathcal{X},\mathcal{P})\big)\\
&\qquad=\ell_2\Big(\psi_2^{\rm cov}\big((\xi,\rho)\,,\,(e_1,\omega_1)\big)\,,\,\psi_2^{\rm cov}\big((e_2,\omega_2)\,,\,(\mathcal{X},\mathcal{P})\big)\Big)\\
&\qquad\quad+\ell_2\Big(\psi_2^{\rm cov}\big((\xi,\rho)\,,\,(e_2,\omega_2)\big)\,,\,\psi_2^{\rm cov}\big((e_1,\omega_1)\,,\,(\mathcal{X},\mathcal{P})\big)\Big)\ .
\end{aligned}
$$

Expanding the left-hand side, we get

$$
\begin{aligned}
\psi_2^{\rm cov}\big((\iota_\xi\omega_1\wedge e_2+\iota_\xi\omega_2\wedge e_1,[\iota_\xi\omega_1,\omega_2]+[\iota_\xi\omega_2,\omega_1])\,,\,(\mathcal{X},\mathcal{P})\big)& \\
+\Big(\mathrm{d}x^\mu\otimes\mathrm{Tr}\big((\iota_\mu[\iota_\xi\omega_1,\omega_2]+\iota_\mu[\iota_\xi\omega_2,\omega_1])\curlywedge\mathcal{P}\big)\,,\,0\Big)& \\
=-\Big(\mathrm{d}x^\mu\otimes\mathrm{Tr}\big((\iota_\mu[\iota_\xi\omega_1,\omega_2]+\iota_\mu[\iota_\xi\omega_2,\omega_1])\curlywedge\mathcal{P}\big)\,,\,0\Big)& \\
+\Big(\mathrm{d}x^\mu\otimes\mathrm{Tr}\big((\iota_\mu[\iota_\xi\omega_1,\omega_2]+\iota_\mu[\iota_\xi\omega_2,\omega_1])\curlywedge\mathcal{P}\big)\,,\,0\Big)& \\
=(0,0)\ ,&
\end{aligned}
$$

while the right-hand side expands as

$$
\ell_2\Big(\big(0,-\iota_\xi\omega_1\big)\,,\,\big(-\mathrm{d}x^\mu\otimes\mathrm{Tr}(\iota_\mu\omega_2\curlywedge\mathcal{P}),0\big)\Big)+_{(1\leftrightarrow2)}=(0,0)\ .
$$

This completes the proof that the maps $\{\psi_n^{\rm cov}\}$ from Section 5.4 indeed do define an L_∞-morphism (in the case $d=3$).

B Calculations in four dimensions

In this appendix we illustrate the explicit dualization of the BV–BRST formalism, focusing on the case $d=4$. That is, we will show that the brackets defined in Section 8.2 are dual to the non-covariant BV differential of [24]. We have already done this in Section 6.1 for the kinematical sector of the Einstein–Cartan–Palatini theory in any dimension d, so we only need to check that the BV transformations of the antifields, given in (6.8), dualize to the remaining brackets of the dynamical sector and those on the space of Noether identities. This may be seen as an alternative proof of the $d=4$ homotopy relations by appealing to the duality with the BV–BRST formalism from Section 6, where they are automatically guaranteed to hold by nilpotency of the BV differential $Q_{\rm BV}^2=0$. Again we shall set $\Lambda=0$ for brevity in these calculations.

Dynamical brackets

We start from the first transformation of (6.8) specialised to the case $d=4$:

$$
Q_{\rm BV}e^{\dagger\,a_1a_2a_3}_{\ \mu_1\mu_2\mu_3}=-e^{[a_1}_{[\mu_1}\,R^{a_2a_3]}_{\mu_2\mu_3]}+4\big(e^{\dagger\,a_1a_2a_3}_{\ [\mu_1\mu_2\mu_3}\,\partial_{\sigma]}\xi^\sigma-e^{\dagger\,[a_1a_2a_3}_{\ \mu_1\mu_2\mu_3}\,\rho^{d]}{}_d\big)+\partial_\sigma\big(\xi^\sigma\,e^{\dagger\,a_1a_2a_3}_{\ \mu_1\mu_2\mu_3}\big)\ . \tag{B.1}
$$

Dualizing we retrieve the dynamical brackets of our L_∞-algebra for the coframe field e in four dimensions, as we now demonstrate.

For $(e,\omega)\in\mathscr{F}_{\rm BV\,0}$, using the natural duality pairing $\langle-|-\rangle$ between $\odot^\bullet\mathscr{F}_{\rm BV}$ and $\odot^\bullet_{\mathbb{R}}\mathscr{F}^\star_{\rm BV}$ we obtain

$$
\begin{aligned}
\langle Q_{\rm BV}e'^{\dagger\,a_1a_2a_3}_{\ \mu_1\mu_2\mu_3}|e\odot\omega\rangle&=\langle-e'^{[a_1}_{[\mu_1}\odot\partial_{\mu_2}\omega'^{a_2a_3]}_{\mu_3]}|e\odot\omega\rangle\\
&=-e^{[a_1}_{[\mu_1}\,\partial_{\mu_2}\omega^{a_2a_3]}_{\mu_3]}\\
&=\langle e'^{\dagger\,a_1a_2a_3}_{\ \mu_1\mu_2\mu_3}|-e\curlywedge\mathrm{d}\omega\rangle\\
&=:(-1)^{|Q_{\rm BV}|\,|e'^{\dagger}|}\,\langle e'^{\dagger\,a_1a_2a_3}_{\ \mu_1\mu_2\mu_3}|\mathrm{D}_{\rm BV\,2}(e\odot\omega)\rangle\ ,
\end{aligned}
$$

where

$$\mathrm{D}_{\mathrm{BV}} := Q^{\star}_{\mathrm{BV}} : \odot^{\bullet}\mathscr{F}_{\mathrm{BV}} \longrightarrow \odot^{\bullet}\mathscr{F}_{\mathrm{BV}}$$

is determined by the decomposition

$$\mathrm{pr}_{\mathscr{F}_{\mathrm{BV}}} \circ \mathrm{D}_{\mathrm{BV}} = \sum_{n=1}^{\infty} \mathrm{D}_{\mathrm{BV}\,n}$$

with component maps $\mathrm{D}_{\mathrm{BV}\,n} : \odot^n\mathscr{F}_{\mathrm{BV}} \to \mathscr{F}_{\mathrm{BV}}$. Thus $\mathrm{D}_{\mathrm{BV}\,2}(e \odot \omega) = e \curlywedge \mathrm{d}\omega$, and so

$$\begin{aligned}
\ell_2({}^{s^{-1}}e \wedge {}^{s^{-1}}\omega) &= s^{-1} \circ \mathrm{D}_{\mathrm{BV}\,2} \circ (s \otimes s)({}^{s^{-1}}e \wedge {}^{s^{-1}}\omega) \\
&= (-1)^{|{}^{s^{-1}}e|}\, s^{-1} \circ \mathrm{D}_{\mathrm{BV}\,2}(e \odot \omega) \\
&= -{}^{s^{-1}}e \curlywedge \mathrm{d}{}^{s^{-1}}\omega
\end{aligned}$$

as required.

Similarly, for $e, \omega_1, \omega_2 \in \mathscr{F}_{\mathrm{BV}\,0}$ we get

$$\begin{aligned}
\langle Q_{\mathrm{BV}} e'^{\dagger}{}^{a_1a_2a_3}_{\mu_1\mu_2\mu_3} | e \odot \omega_1 \odot \omega_2 \rangle &= \langle -e'^{[a_1}_{[\mu_1} \odot \omega'^{a_2}{}_{|c|\mu_2} \odot \omega'^{|c|a_3]}_{\mu_3]} | e \odot \omega_1 \odot \omega_2 \rangle \\
&= -\big(e^{[a_1}_{[\mu_1}\, \omega_1{}^{a_2}{}_{|c|\mu_2}\, \omega_2{}^{|c|a_3]}_{\mu_3]} + e^{[a_1}_{[\mu_1}\, \omega_2{}^{a_2}{}_{|c|\mu_2}\, \omega_1{}^{|c|a_3]}_{\mu_3]} \big) \\
&= \langle e'^{\dagger}{}^{a_1a_2a_3}_{\mu_1\mu_2\mu_3} | - e \curlywedge [\omega_1, \omega_2] \rangle \\
&=: \langle e'^{\dagger}{}^{a_1a_2a_3}_{\mu_1\mu_2\mu_3} | - (-1)^{|Q_{\mathrm{BV}}|\,|e'^{\dagger}|}\, \mathrm{D}_{\mathrm{BV}\,3}(e \odot \omega_1 \odot \omega_2) \rangle \ .
\end{aligned}$$

Thus $\mathrm{D}_{\mathrm{BV}\,3}(e \odot \omega_1 \odot \omega_2) = e \curlywedge [\omega_1, \omega_2]$, and so

$$\begin{aligned}
\ell_3({}^{s^{-1}}e \wedge {}^{s^{-1}}\omega_1 \wedge {}^{s^{-1}}\omega_2) &= s^{-1} \circ \mathrm{D}_{\mathrm{BV}\,3} \circ (s \otimes s \otimes s)({}^{s^{-1}}e \wedge {}^{s^{-1}}\omega_1 \wedge {}^{s^{-1}}\omega_2) \\
&= (-1)^{2\,|{}^{s^{-1}}e| + |{}^{s^{-1}}\omega_1|}\, \big({}^{s^{-1}}e \curlywedge [{}^{s^{-1}}\omega_1, {}^{s^{-1}}\omega_2]\big) \\
&= -{}^{s^{-1}}e \curlywedge [{}^{s^{-1}}\omega_1, {}^{s^{-1}}\omega_2]
\end{aligned}$$

as required.

Next, we note that the Lie derivative appears explicitly in (B.1), as the fourth and second terms expand into

$$\begin{aligned}
&\partial_\sigma\xi^\sigma\, e^{\dagger}{}^{a_1a_2a_3}_{\mu_1\mu_2\mu_3} + \xi^\sigma\, \partial_\sigma e^{\dagger}{}^{a_1a_2a_3}_{\mu_1\mu_2\mu_3} + e^{\dagger}{}^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}\, \partial_\sigma\xi^\sigma - e^{\dagger}{}^{a_1a_2a_3}_{[\mu_1\mu_2|\sigma|}\, \partial_{\mu_3]}\xi^\sigma + e^{\dagger}{}^{a_1a_2a_3}_{[\mu_1|\sigma|\mu_2}\, \partial_{\mu_3]}\xi^\sigma - e^{\dagger}{}^{a_1a_2a_3}_{\sigma[\mu_1\mu_2}\, \partial_{\mu_3]}\xi^\sigma \\
&\qquad = \xi^\sigma\, \partial_\sigma e^{\dagger}{}^{a_1a_2a_3}_{\mu_1\mu_2\mu_3} + \partial_{[\mu_3}\xi^\sigma\, e^{\dagger}{}^{a_1a_2a_3}_{\mu_1\mu_2]\sigma} + \partial_{[\mu_2}\xi^\sigma\, e^{\dagger}{}^{a_1a_2a_3}_{\mu_1|\sigma|\mu_3]} + \partial_{[\mu_1}\xi^\sigma\, e^{\dagger}{}^{a_1a_2a_3}_{|\sigma|\mu_2\mu_3]}
\end{aligned}$$

where we used $|\xi| = 1$ and $|e^\dagger| = -1$. This expression extracts the components of the Lie derivative of a three-form by dualization, as expected. Explicitly, for $\xi \in \mathscr{F}_{\mathrm{BV}\,-1}$ and $e^\dagger \in \mathscr{F}_{\mathrm{BV}\,1}$ we get

$$\begin{aligned}
&\langle Q_{\mathrm{BV}} e'^{\dagger}{}^{a_1a_2a_3}_{\mu_1\mu_2\mu_3} | \xi \odot e^\dagger \rangle \\
&\quad = \langle \xi'^\sigma \odot \partial_\sigma e'^{\dagger}{}^{a_1a_2a_3}_{\mu_1\mu_2\mu_3} + \partial_{[\mu_3}\xi'^\sigma \odot e'^{\dagger}{}^{a_1a_2a_3}_{\mu_1\mu_2]\sigma} + \partial_{[\mu_2}\xi'^\sigma \odot e'^{\dagger}{}^{a_1a_2a_3}_{\mu_1|\sigma|\mu_3]} + \partial_{[\mu_1}\xi'^\sigma \odot e'^{\dagger}{}^{a_1a_2a_3}_{|\sigma|\mu_2\mu_3]} | \xi \odot e^\dagger \rangle \\
&\quad = (-1)^{|\xi'|\,|e'^\dagger|}\, (\mathrm{L}_\xi e^\dagger)^{a_1a_2a_3}_{\mu_1\mu_2\mu_3} \\
&\quad = \langle e'^{\dagger}{}^{a_1a_2a_3}_{\mu_1\mu_2\mu_3} | \mathrm{L}_\xi e^\dagger \rangle \\
&\quad =: -(-1)^{|Q_{\mathrm{BV}}|\,|e'^\dagger|}\, \langle e'^{\dagger}{}^{a_1a_2a_3}_{\mu_1\mu_2\mu_3} | \mathrm{D}_{\mathrm{BV}\,2}(\xi \odot e^\dagger) \rangle \ .
\end{aligned}$$

Thus $\mathrm{D}_{\mathrm{BV}\ 2}(\xi \odot e^\dagger) = \mathrm{L}_\xi e^\dagger$, and so

$$\ell_2({}^{s^{-1}}\xi \wedge {}^{s^{-1}}e^\dagger) = s^{-1} \circ \mathrm{D}_{\mathrm{BV}\ 2} \circ (s \otimes s)({}^{s^{-1}}\xi \wedge {}^{s^{-1}}e^\dagger) = \mathrm{L}_{{}^{s^{-1}}\xi}\, {}^{s^{-1}}e^\dagger$$

as required.

Lastly, we note that the part concerning the local Lorentz transformations in (B.1) may be expanded as

$$4\, \rho^{[d}{}_d\, e^{\dagger\, a_1 a_2 a_3]}{}_{\mu_1\mu_2\mu_3} = -\rho^{[a_1}{}_d\, e^{\dagger |d| a_2 a_3]}{}_{\mu_1\mu_2\mu_3} - \rho^{[a_2}{}_d\, e^{\dagger\, a_1 |d| a_3]}{}_{\mu_1\mu_2\mu_3} - \rho^{[a_3}{}_d\, e^{\dagger\, a_1 a_2] d}{}_{\mu_1\mu_2\mu_3}$$

where we used antisymmetry of ρ^{ab}. This contains the action of an infinitesimal Lorentz transformation on a three-vector: Indeed, the corresponding Euler–Lagrange derivative transforms as

$$\rho \cdot (e \curlywedge R) = \rho \cdot e \curlywedge R + e \curlywedge [\rho, R] = \big(\rho^a{}_d\, e^d \wedge R^{bc} + e^a \wedge \rho^b{}_d\, R^{dc} + e^a \wedge \rho^c{}_d\, R^{bd}\big)\, \mathtt{E}_a \wedge \mathtt{E}_b \wedge \mathtt{E}_c\ .$$

Dualizing for $\rho \in \mathscr{F}_{\mathrm{BV}\ -1}$ and $e^\dagger \in \mathscr{F}_{\mathrm{BV}\ 1}$ we obtain

$$\begin{aligned}
\langle Q_{\mathrm{BV}} e'^{\dagger\, a_1a_2a_3}{}_{\mu_1\mu_2\mu_3} | \rho \odot e^\dagger\rangle &= \langle -\rho'^{\,[a_1}{}_d \odot e'^{\dagger |d| a_2a_3]}{}_{\mu_1\mu_2\mu_3} - \rho'^{\,[a_2}{}_d \odot e'^{\dagger\, a_1|d|a_3]}{}_{\mu_1\mu_2\mu_3} - \rho'^{\,[a_3}{}_d \odot e'^{\dagger\, a_1a_2]d}{}_{\mu_1\mu_2\mu_3} | \rho \odot e^\dagger\rangle \\
&= -(-1)^{|\rho'|\,|e'^\dagger|}\, \big(\rho^{[a_1}{}_d\, e^{\dagger|d|a_2a_3]}{}_{\mu_1\mu_2\mu_3} - \rho^{[a_2}{}_d\, e^{\dagger\, a_1|d|a_3]}{}_{\mu_1\mu_2\mu_3} - \rho^{[a_3}{}_d\, e^{\dagger\, a_1a_2]d}{}_{\mu_1\mu_2\mu_3}\big) \\
&= \langle e'^{\dagger\, a_1a_2a_3}{}_{\mu_1\mu_2\mu_3} | \rho \cdot e^\dagger\rangle \\
&=: (-1)^{|Q_{\mathrm{BV}}|\,|e'^\dagger|}\, \langle e'^{\dagger\, a_1a_2a_3}{}_{\mu_1\mu_2\mu_3} | \mathrm{D}_{\mathrm{BV}\ 2}(\rho \odot e^\dagger)\rangle\ .
\end{aligned}$$

Thus $\mathrm{D}_{\mathrm{BV}\ 2}(\rho \odot e^\dagger) = -\rho \cdot e^\dagger$ and so

$$\ell_2({}^{s^{-1}}\rho \wedge {}^{s^{-1}}e^\dagger) = s^{-1} \circ \mathrm{D}_{\mathrm{BV}\ 2} \circ (s \otimes s)({}^{s^{-1}}\rho \wedge {}^{s^{-1}}e^\dagger) = -{}^{s^{-1}}\rho \cdot {}^{s^{-1}}e^\dagger$$

as desired. Hence we have recovered the full dynamical L_∞-algebra for the coframe field e given in Section 8.2. One similarly obtains the dynamical brackets for the connection ω from the second transformation in (6.8).

Noether identity brackets

Now we consider the third transformation of (6.8) specialised to $d = 4$ dimensions:

$$\begin{aligned}
Q_{\mathrm{BV}} \rho^{\dagger\, a_1a_2}{}_{\mu_1\mu_2\mu_3\mu_4} = &- \tfrac{3}{2}\, e_{d[\mu_1}\, e^{\dagger\, da_1a_2}{}_{\mu_2\mu_3\mu_4]} - \omega^{a_1}{}_{d[\mu_1}\, \omega^{\dagger\, da_2}{}_{\mu_2\mu_3\mu_4]} + \partial_{[\mu_1} \omega^{\dagger\, a_1a_2}{}_{\mu_2\mu_3\mu_4]} \\
&- \rho^{a_1}{}_d\, \rho^{\dagger\, da_2}{}_{\mu_1\mu_2\mu_3\mu_4} + \partial_\sigma\big(\xi^\sigma\, \rho^{\dagger\, a_1a_2}{}_{\mu_1\mu_2\mu_3\mu_4}\big)\ .
\end{aligned} \tag{B.2}$$

The first three terms extract the components of the brackets corresponding to the Noether identity for local $\mathsf{SO}_+(1,3)$ Lorentz transformations:

$$-\mathrm{d}^\omega \mathcal{F}_\omega - \tfrac{3}{2}\, \mathcal{F}_e \wedge e = 0\ , \tag{B.3}$$

as we now demonstrate.

Dualizing as we did previously, for $\omega^\dagger \in \mathscr{F}_{\mathrm{BV}\ 1}$ we get

$$\langle Q_{\mathrm{BV}} \rho'^{\dagger\, a_1a_2}{}_{\mu_1\mu_2\mu_3\mu_4} | \omega^\dagger\rangle = \partial_{[\mu_1} \omega^\dagger_{\mu_2\mu_3\mu_4]} = \langle \rho'^{\dagger\, a_1a_2}{}_{\mu_1\mu_2\mu_3\mu_4} | \mathrm{d}\omega^\dagger\rangle =: \langle \rho'^{\dagger\, a_1a_2}{}_{\mu_1\mu_2\mu_3\mu_4} | \mathrm{D}_{\mathrm{BV}\ 1}\omega^\dagger\rangle$$

where we used $|Q_{\mathrm{BV}}| = 1$ and $|\omega^\dagger| = 2$. Thus $\mathrm{D}_{\mathrm{BV}\ 1}\omega^\dagger = \mathrm{d}\omega^\dagger$ and so

$$\ell_1({}^{s^{-1}}\omega^\dagger) = s^{-1} \circ \mathrm{D}_{\mathrm{BV}\ 1} \circ s({}^{s^{-1}}\omega^\dagger) = \mathrm{d}\, {}^{s^{-1}}\omega^\dagger\ .$$

Next, for $e\in\mathscr{F}_{\mathrm{BV}\,0}$ and $e^\dagger\in\mathscr{F}_{\mathrm{BV}\,1}$ we obtain

$$
\begin{aligned}
\langle Q_{\mathrm{BV}}\rho'^{\dagger}{}^{a_1a_2}_{\mu_1\mu_2\mu_3\mu_4}|e\odot e^\dagger\rangle &= -\tfrac{3}{2}\,e_{d[\mu_1}\,e^{\dagger\,da_1a_2}_{\mu_2\mu_3\mu_4]} \\
&= -\tfrac{3}{2}\,(e^\dagger\wedge e)^{a_1a_2}_{\mu_1\mu_2\mu_3\mu_4} \\
&= \langle \rho'^{\dagger}{}^{a_1a_2}_{\mu_1\mu_2\mu_3\mu_4}|-\tfrac{3}{2}\,e^\dagger\wedge e\rangle \\
&=: \langle \rho'^{\dagger}{}^{a_1a_2}_{\mu_1\mu_2\mu_3\mu_4}|\mathrm{D}_{\mathrm{BV}\,2}(e\odot e^\dagger)\rangle \ .
\end{aligned}
$$

Thus $\mathrm{D}_{\mathrm{BV}\,2}(e\odot e^\dagger)=-\frac{3}{2}\,e^\dagger\wedge e$ and so

$$
\ell_2({}^{s^{-1}}e\wedge{}^{s^{-1}}e^\dagger)=s^{-1}\circ\mathrm{D}_{\mathrm{BV}\,2}\circ(s\otimes s)({}^{s^{-1}}e\wedge{}^{s^{-1}}e^\dagger)=-s^{-1}\circ\mathrm{D}_{\mathrm{BV}\,2}(e\odot e^\dagger)=\tfrac{3}{2}\,{}^{s^{-1}}e^\dagger\wedge{}^{s^{-1}}e \ .
$$

Continuing in an identical fashion, for $\omega\in\mathscr{F}_{\mathrm{BV}\,0}$ and $\omega^\dagger\in\mathscr{F}_{\mathrm{BV}\,1}$ we find

$$
\ell_2({}^{s^{-1}}\omega\wedge{}^{s^{-1}}\omega^\dagger)={}^{s^{-1}}\omega\wedge{}^{s^{-1}}\omega^\dagger \ .
$$

The Noether identity (B.3) is then encoded in $Q_{\mathrm{BV}}^2\rho^\dagger=0$.

It is easy to see that the fourth term in (B.2) dualizes to the action of a local Lorentz transformation on a two-vector: For $\rho\in\mathscr{F}_{\mathrm{BV}\,-1}$ and $\rho^\dagger\in\mathscr{F}_{\mathrm{BV}\,2}$, via similar manipulations we find

$$
\ell_2({}^{s^{-1}}\rho\wedge{}^{s^{-1}}\rho^\dagger)=-{}^{s^{-1}}\rho\cdot{}^{s^{-1}}\rho^\dagger \ .
$$

Lastly, the fifth term in (B.2) extracts the components of the Lie derivative of a four-form: For $\xi\in\mathscr{F}_{\mathrm{BV}\,-1}$ and $\rho^\dagger\in\mathscr{F}_{\mathrm{BV}\,2}$ we obtain

$$
\langle Q_{\mathrm{BV}}\rho'^{\dagger}{}^{a_1a_2}_{\mu_1\mu_2\mu_3\mu_4}|\xi\odot\rho^\dagger\rangle=\partial_\sigma\big(\xi^\sigma\,\rho^{\dagger}{}^{a_1a_2}_{\mu_1\mu_2\mu_3\mu_4}\big)=\langle\rho'^{\dagger}{}^{a_1a_2}_{\mu_1\mu_2\mu_3\mu_4}|\mathrm{L}_\xi\rho^\dagger\rangle=:\langle\rho'^{\dagger}{}^{a_1a_2}_{\mu_1\mu_2\mu_3\mu_4}|\mathrm{D}_{\mathrm{BV}\,2}(\xi\odot\rho^\dagger)\rangle
$$

and so

$$
\ell_2({}^{s^{-1}}\xi\wedge{}^{s^{-1}}\rho^\dagger)=s^{-1}\circ\mathrm{D}_{\mathrm{BV}\,2}\circ(s\otimes s)({}^{s^{-1}}\xi\wedge{}^{s^{-1}}\rho^\dagger)=\mathrm{L}_{{}^{s^{-1}}\xi}\,{}^{s^{-1}}\rho^\dagger \ .
$$

These thus recover the actions of local Lorentz transformations and diffeomorphisms on the Noether identities corresponding to the $\mathsf{SO}_+(1,3)$ gauge symmetry. One similarly obtains the brackets for the Noether identities corresponding to diffeomorphisms, together with the action of gauge transformations on them, from the fourth transformation in (6.8).

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